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ACTION of INFORMATION

energy, time and communication

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Translations into English from the original Serbian language are intended for my private use, to test the meaning of sentences. But public use is completely allowed to others. The author.

RASTKO VUKOVIĆ
ACTION OF INFORMATION – ENERGY, TIME AND COMMUNICATION

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I WROTE THE TEXT FROM DECEMBER 2019 TO SEPTEMBER 2020.

ROUGH TRANSLATION FROM SERBIAN:

РАСТКО ВУКОВИЋ
ДЕЈСТВО ИНФОРМАЦИЈЕ – ЕНЕРГИЈА, ВРЕМЕ И КОМУНИКАЦИЈА

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Рецензенти:
ДУШКО МИЛИНЧИЋ, ПРОФ. ИНФОРМАТИКЕ
АЛЕКСАНДРА РАДИЋ, ПРОФ. ФИЗИКЕ

Preface

It seems that you the newer ideas deliberately published in worse places — a colleague recently asked me — isn't that a little weird?

What else but to laugh — I said — and you are not far from the truth! Educated people are trained to spin foreign and alleged knowledge. They are hard on originality, and the originality is hard. The places of rating and institutions are their field, so maybe I'm not wrong when refraining from such.

The text that follows is almost identical left "unnoticed" on the Internet (Scribid) in December 2019, and its popular parts could be published in a newspaper column (portal `http://izvor.ba/`) from week to week months later. I was allowed to do that, expecting that those to whom these texts concern trust the officials more than themselves. There is a well-known "secret" of modern education systems, the failure in creativity and overthrow of the alleged intelligence of their products.

The value of knowledge itself and those who learn it should not be underestimated and that is it. They are the goal of these discussions, only the timing has changed a bit. I do not expect these texts to be understood at the moment, and when (if) they are, I hope that they will not become an ideology. It would be disappointing for a theory of the world whose essence is information and information whose essence is unpredictability, that is, an anti-dogma theory, to become a dogma.

Author, May 2020.

Sadržaj

Glava 1

Popular Stories

Introduction

As early as December 2019, it was clear that I had too many topics for "Minimalism of Information", book [2], but also that the surplus would not fit under the cap of "Action of Information". It was estimated that about 20 popular stories of the information saved at that time could be published in the weekly publications of the column "izvor.ba" until around May 2020, and that in the meantime so many more will appear, and so they would.

Although the "information theory" itself (mine) is in principle very simple, it is based on the uncertainty of some natural phenomena and includes the laws of conservation and minimalism of the "amount of data", it turns out to be too broad for physics itself. I would be sorry not to point out its consequences in social phenomena, and I would not like to miss the "informatics" explanation of some of today's supposedly inexplicable phenomena of physics, first of all quantum, then cosmological, and then classical. Then there are the issues of revisions of perhaps some misconceptions of science, so that the "central" topics (which I had in mind at the beginning) never become current.

This is a book about the effect of information conceived in the confrontation of the concepts of energy and time with the principles of uncertainty, with the backbone of, say, the Higgs boson and other fields of the so-called basic forces of physics, but with the development of these popular stories and their more or less follow-up with mathematics, added in the second chapter, it raised into an introduction of the introduction to its main ideas. Although I am only writing the 18th story as I write this, it is clear to me that there will be so many more in the book and that I will not reach the given point.

It would be a pity to miss the popular descriptions of important by-results of the new theory, especially when the gap between the exact and social sciences is widening. It is common today to philosophize that mathematics has nothing on the path of social phenomena. I would ask, then, is informatics something that could not bothered by the social, and is the Kolmogorov probability a heresy in mathematics?

A different kind of problem comes from the sciences themselves. It is interesting, for example, that the definition of entropy ($S = Q/T$) as we take it from Clausius' expression (1855) shows less good than "generalized entropy" from my contributions, which are supposed to be a hypothetical and speculative idea based on "principles of information". Knowledge can have both enemies and allies in delusion; it is to believe that the victory of truth comes with thorough preparation and setting without haste, before all, because its discovery is always a surprise.

1.1 Life cycle

The rise and fall of each of about thirty known leading civilizations was accompanied, among other things, by the development of some form of the *legal* system.

At the beginning, in the phase of youth, the society has fresh norms of behavior which are supplemented over time by developing and stabilizing the system. After maturity, they become more complicated as the community slows down and becomes obsolete. The time of old age is when judicial repairs do not help with additional restrictions as they used to. A similar concept applies to many *life cycles* – from firms to living things – and easily supervene on information theory.

I will not cite typical examples here of *civilization* (as before) so that deduction from history and statistics would not mislead us. With the discovery, I go the other way – the theory recognizes the facts, and if it is good, it will redefine those that "do not fit" and predict exceptions. Nothing below that.

Great civilization was ancient China two or more centuries before the new era – according to the inventions of compasses and gunpowder, making paper and printing, observing comets and eclipses of the Sun. They measured time with shadows, developed weapons (crossbows) and became a safer and more orderly state from within.

Researchers of the pre-imperial and imperial period of China are familiar with the tradition of growing protection of personal rights, property, contracts, family relations and heritage. It is reminiscent of the Roman Republic and its later imperial period of enrichment of the Law of the Twelve Tables, in the first period unchanging.

The ancient Egyptians were especially successful in mathematics, architecture and medicine. They came far ahead of their time in terms of gender equality – men and women (except slaves) were considered equal before the law. Of the approximately 170 pharaohs, six were women, the first to Sobekneferu, and the last to Cleopatra.

Their legal system is based on the "harmony of the beginning of time" (ma'at) – that one should be at peace with oneself, society and the gods, and violators are often cruelly punished. The judges, as today, were people considered experts in the field, the courts weighed the findings of the offense, and the police forcibly apprehended the offenders, but at the top was the king (Pharaoh). Earlier regulations were simpler to multiply later, resulting in growing bureaucracy, excessive false testimony, and a loss of faith in the concept.

The Inca civilization, in the place of today's Peru, built sophisticated and extensive roads, as well as very strict and sharp laws, starting from three basic sets: "Ama Sua, Ama Llulla, Ama Quella" (do not steal, do not lie, do not be lazy). These and the laws for maintaining a moral and disciplined society in the Inca Empire tightened social stability. The Inca government promoted peace among its citizens, and when any crime was committed, the punishments were relentless.

The point of this story will not be disputed by worse examples than the above, nor by the cases of the cessation of civilization due to an accident. The main thing is that from the stinginess of the action, a surplus can arise, and from the surplus a life, during which risk and order interchange.

Information is *quantity of possibilities*, which makes it close to the concepts of freedom and rights. The ultimate aspects of this measure are uncertainties and outcomes, and the principles are maintenance (conservation) and savings. Inertia and the principle of minimalism bring information in connection with physical action. Some of their consequences are well known to physics (reflection and refraction of light, movement of trajectories in the field of force), others are less known (spontaneous growth of entropy), and here targeted (striving for security, inaction and strengthening the rule of law) are unknown as such.

The law forms restrictions. By subtracting some options, the probabilities of others increase, which directs us. The attractiveness of run-in is reminiscent of railways, trains running on rails for controlled attention and acting devoid of the need for painful surprises. The rule of law rewards us with security and efficiency and a “sweet” feeling of living without risk, of sudden changes in energy, force and aggression. Seeking protection from the state, we actually strive for a state of less creativity, less responsibility; we hand over our freedoms to it “for safekeeping”. By conveying our vitality and intelligence through regulations, we delude ourselves that the environment can stand or that our improved organization can overcome it.

However, the legal organism becomes obsolete as it matures. The environment takes advantage of the options forbidden to us and goes further in ways unimaginable to a rigid body. In the nature of information is *unpredictability* whose power we see as the originality and audacity of intruders.

With better-established bans, communities are sinking into their *couplings* because they are more likely (fidelity). The more probable happens more often, it carries less information and has less action, so associations “spontaneously” evolve towards such. It becomes the glue of association. That move into order, in a freer environment, makes it a victim. Once a great civilization eventually becomes a restricted, a pray of the brave or a patient in need of outside help.

The rigid old system does not recognize or see other optimums of free options. As it progresses in the dictatorship phase, with diminished confidence in truth and freedom, it is stubbornly blind to multitudes like the different biological species that are equally well adapted to the same natural environment. Many *biological species* are particularly attached to the common ground by their own perceptions. Further accumulation of bans and state coercion is followed by disintegration. However, they are artificial and unnatural, they defy the truth, that is why they are filled with mistakes and alienated from the initial ideals.

http://izvor.ba/
January 31, 2020.

1.2 Stockholm Syndrome

Stockholm Syndrome is a psychological state of connection between abductees and kidnappers. The term was first used by *Bejerot*[1]. The hostages became emotionally attached to the robbers and then justified their actions. Later, at the time of the trial, they were reluctant to talk about the incident.

Let us ignore for now all other known and unknown explanations, and consider similar states or processes from the point of view of information perception and probability. Suppose we have two subjects, a person and a situation, with a series of events that we each value in our own way.

Values are, for example, a person’s ability to deal with a given event and the situation’s ability to direct / limit it. The product of ability and the corresponding restriction over the same perception is freedom, and the sum of all freedoms is the information of perception.

Analogous to this is the “coupling of chances”. Both are scalar products of vectors in the same event space and therefore with the same number of components — the first information,

[1] Nils Bejerot (1921-1988), Swedish psychiatrist and criminologist.

and the second the probability of the outcome. *Product distribution* probability is a number, from zero to one, the largest when the larger component of the first is multiplied by the larger of the second, or smaller with the smaller. In contrast, in the growing series of components of the first vector, and the declining ones of the second, the product is minimal.

Information is the logarithm of probability, and the couplings of chance as well as perception have a common form and all the more interesting interpretations. The larger the *scalar product*, the larger the coupling, means the more aligned the vectors. They are then "more parallel", the abilities are more focused on limitations. Better adaptation of persons to the environment seemingly increases overall freedom; the exclusion of the part of possibility focuses us on what we can and frees us from excess effort.

To understand this even better, let us remember that information is an action and that the principle of minimalism applies to physical actions. We avoid the unknown not only because of fear, perhaps innate, but also because of resistance to effort, and that is why we love order and security, and we easily confuse efficiency with creativity. On the other hand, uncertainties and environments are inevitable.

Whatever conditions the subject is in, it is somewhere. Abilities are plastic and will try to adapt to some of the offered limitations, in one way or another, with this or that secondary reason behind which actually stands the principled tendency of nature to realize greater probability, i.e. less (real) information.

When we have power we will talk around about peace to gain more power knowing that everyone is a prisoner of some hierarchies. We are bound by the illusion of freedom to choose what will dominate us, and which is often reduced to choosing to avoid choosing.

The lives of micro-particles are no different. We interpret them equally using the previous "freedoms" or their exponents which are "chances". In quantum mechanics, chances define the superposition of a given quantum state (particle), we can say the probability distribution. The two quantum states are coupled whose intensity (state alignment) measures the scalar product of their distributions. This product takes probability values, from zero to one, with the higher value being the more probable coupling.

No matter what quantum state we are talking about, it is always in some environment. The same applies to processes that are also vectors of the so-called dual space states. The tendency of states (processes) to associate grows with the scalar product, and their "selectivity" is partly deterministic and partly stochastic, first because of the necessity of the theorems, and second because of the randomness of the phenomena to which they refer.

It's unbelievable, but in the action of information there is the same deep cause that makes us subordinates, because of which lies spread faster than the truth on the Internet, which makes it easier to encode than decode, or which keeps the Earth in orbit around the Sun and an electron bound in an atom.

`http://izvor.ba/`
February 7, 2020.

1.3 Uncertainty Relations

Heisenberg's[2] relations of uncertainty (1927) are one of the most important discoveries in quantum mechanics and perhaps one of the most influential recent ideas.

[2] Werner Heisenberg (1901-1976), German theoretical physicist.

They were discovered by "observing" a particle through an imaginary microscope, for which we take photons (light, electromagnetic radiation) of shorter wavelengths to determine its position more precisely, but therefore with a higher momentum and the transmission of more of its uncertainty into a collision. The calculation shows that the order of magnitude of the product of the indeterminacy of the position and momentum of the observed particle is not less than the *Planck*[3] constant, the *quantum* of action.

Light (electromagnetic wave) has energy which, multiplied by the duration of one oscillation, gives Planck's constant, so it is of the same order the product of magnitudes of energy and duration. In space-time, the three momentum and energy form four coordinates, corresponding to the length, width, height, and time required for light to travel a certain distance.

The connection between quantum action and information is illustrated by a well-known example of *digital records* image and film. In a given magnetic memory, the image is sharper the times the movement is less detailed. The detail of the image (pixel) is proportional to the length, and the momentum to the speed.

It is absurd, but the deeper causes of uncertainty lay in certainty, more precisely said, the dependency hang on random events. It is not possible to change the momentum of a particle without changing its position and vice versa and there is also no instantaneous change in energy nor can time flow without its exchange. This further means that the momentum and position are dependent quantities; they complement each other in such a way that the change of the first and the second is not equal to the change of the second and the first.

Dependent processes are represented by *non commutative operators*, independent by commutative ones. Quantum evolution follow special linear functions for which we know from algebra that their compositions are usually not commutative. For example, doubling the number and adding a unit: doubling the number three and increasing it by one gives seven, but three times increasing by one and then doubling gives eight. Dependent quantum processes and only they have a similar non commutative mapping.

The equivalent of copying a vector (by a unitary) operator is a change in the quantum state. In Heisenberg's dependent quantum development, a change in momentum and then a change in position give a different result than a change in position and then a momentum. The difference between the two compositions corresponds to the quantum of action, now we say the quantum of information. Some of these secrets were revealed during the 20th century.

When we work with non commutative operators of quantum physics in general, then we are talking about the *uncertainty principle*. The difference that would result from a different order of activity of two dependent processes is some *indivisible* magnitude. This is again in accordance with the finite divisibility of each property of information, and that with the law of conservation of information, as opposed to infinite sets which can be their own (proper) real parts.

These are the deeper causes of quantization of information and action. That is why all legal regulations, rules of normal games, laws of social and natural sciences as well as mathematics are always discrete sets. Thus, we are already in my appendix to quantum theory.

If we insist that action is a product of momentum and length (or energy and time), that information transmits action, and that it is the basic and only component of space, time and matter, then we claim that any pair of non commutative physical operators can be reduced

[3]Max Planck (1858-1947), German theoretical physicist.

to a type of mapping position and momentum. Eventually, the notion of "information" is more complex than it seems at first glance, or that I have exaggerated somewhere in these generalizations.

In any case, states and processes are always some "particles", because operators are also vectors dual to the states they act on. Consistent with the main above thesis, only those particles that can communicate act on each other, and we join non-commutative operators to pairs of such. The first then belongs to space, the second to substance, which roughly means that they are represented by some kind of operator of "position" and "momentum", each in its own way.

It is strange, but it agrees with the physically known division of *elementary particles* into bosons and fermions. The former are tolerant like photons, they can occupy the same states. Others are as intolerant as the electrons to which the *Pauli exclusion principle* applies: two identical fermions cannot be in the same quantum state.

Here bosons do not communicate immediate with bosons or fermions with fermions, which should be checked along with the prediction that only some bosons communicate with some fermions. It is known that some bosons (say photons) build force fields to which the corresponding fermions (electrons, protons) are sensitive.

Such a simple division of elementary particles according to action, into bosons and fermions, given the differences and multiplicities that are in the nature of information itself, tells us that they should be packages, possibly so abstract parts that it is not possible to physically unpack them.

Another direction in the development of the uncertainty principle could go to the information of perception. For example, the abilities of freedom are interpreted by impulsivity, and restrictions by spaciousness. We already consider the wavelength to be an indeterminacy of position, and the only step from there is information.

The smaller the position uncertainty, the more likely it is to find a particle at a given location, and then the smaller the location information. Thus, Heisenberg's relations of uncertainty speak of information of perception, and then dual of multiplication of distributions.

http://izvor.ba/
February 21, 2020.

1.4 Potential Energy

Potential energy is that which an object has in its position relative to another object. To lift an object on the closet, we invest some energy, but we get it back by letting the body fall.

By lifting an object we do the work and that is why we are talking about stored energy. However, the change in energy by duration is an action, a physical action that we now interpret as information. So, we are talking about the potential of physical information.

Two separate magnets when attracted can do the job. Similar to gravity, attraction produces supposedly stored energy, but in fact a deficient potential that is replaced by another in accordance with the law of conservation and the principle of least action. The lack of potential energy is attractive, the excess is repulsive.

According to the *Hooke's*[4] law (1676) the force of a spring is proportional to the elon-

[4]Robert Hooke (1635-1703), English scientist.

gation (compression) from which it follows that the potential energy of a body attached to a spring increases by the square of the distance from zero, equilibrium position. When the spring is released, the excess of that energy is "melted" and replenished with kinetic energy so that the total energy of the spring remains the same.

The kinetic energy, otherwise proportional to half the mass of the body and the square of the velocity, increases to the equilibrium position of the spring where all the initial potential energy becomes kinetic. Due to inertia, the stretching (compression) of the spring continues into compression (stretching), and the speed of the body slows down to the end point where it stops when all the kinetic energy turns into potential. An ideal spring (without friction) is an ideal *harmonic oscillator* with periodic changes of energies which, similar to the periods of rotation of the planets around the Sun, tell us something more.

The gravitational potential of a point is usually defined as the work required bringing a unit of mass from infinity to a given point. I add, the body tends to a state of less information, and accelerates to maintain action. The mass in the gravitational field falls freely, it is weightless for itself, but for others its total energy, kinetic and potential, remains constant.

Kepler's[5] Second Law (1609) says that a line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. There is vector evidence that this is a property of any constant force of a given source whose carriers (bosons) could last like photons or gravitons. On the other hand, using the commutator algebra [6] it can be proved that this surface is equal to the constant action of charge on its way through the force field.

In short, imagine a straight line and a point outside it at a given distance. That point is the source of (zero) force and the segment of the given line is a part of the path of the celestial body (charge) during some time interval. The segment and the source form a triangle of the surface that does not depend on the location of the segment on a given line. The surface is then defined analytically.

The triangle is in a plane with two coordinate axes. The difference of alternating products, the first coordinates of the first vertex of a triangle with the second of the second and the first coordinates of the second with the first of the first, is called the *commutator* of the pair of points. There are three such (oriented) pairs of triangle points and three commutators, and their (semi) sum is the area of the triangle. In general, the sum of the commutators of successive pairs of points of a broken line in a plane is equal to the (semi) surface of the interior that the line encloses. It's a new thing (my) in analytic geometry, but it's easy to check.

When one vertex of a triangle is the origin of a coordinate system, then its double surface is equal to the commutator of the remaining pair of vertices. Therefore, the commutator is a surface measure!

This is easily transcribed into vectors and their operators (vectors dual to the first), when it turns out that the commutator is the value of the product of Heisenberg's famous relations, the product of the uncertainties of position and momentum, i.e. energy and time. It is the smallest of the order of Planck's constant, the quantum of action, so the commutator is then also the smallest physical information.

It's hard to retell these formulas, but I can try. The constant "Kepler surface" is equivalent to information, and it is equivalent to action. All three are two-dimensional. They are

[5]Johannes Kepler (1571-1630), German mathematician and astronomer.

[6]R.V. Potential information `https://www.academia.edu/41986473/`

proportional to the surfaces of concentric (virtual) spheres of bosons by which the source of force speaks to the world about itself, and the law of motion follows from the relation of the mentioned surfaces.

The body moves through the *force field* like an ant over obstacles. They are not spent themselves breaking through the surface, but in their own way follow the shortest paths, now we say trajectories of the least action, that is the least communication. So we also go over the hill when there are no tunnels through the hill.

The consequences of the new point of view are different. For example, *geodesic lines* of non-Euclidean geometry and general theories of relativity are feasible from the principle of least action of physics. The idea of *dark matter* due to the disagreement of theories of gravity and the distribution of masses in galaxies cannot be corrected by simply "fixing" gravity. *Newton's*[7] formula does not follow from Kepler's second law, but from the *third*: the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit. And those are already three or four examples.

There are implications of the above discovery for social phenomena as well. Striving to get rid of excess uncertainty, information that irritates us with the unknown, we surrender our free will to various authorities. By submitting to individuals, groups, the rule of law, we gravitate to order, security, or efficiency. By avoiding unpleasant options, the personal loss of decision-making is complemented by the (not always pleasant) action of the organization to which we surrender.

`http://izvor.ba/`
February 28, 2020.

1.5 Quantum States and Processes

Quantum mechanics have been initiated by the *Heisenberg* discovery of uncertainty relations.

It was preceded by the successes of *Planck* energy quantization in explaining dark body radiation and *Einstein*[8] Photoelectric effect, *Louis de Broglie*[9] particle interpretation by waves, *Schrödinger*[10] equation, and at it all came the discovery of that world as a representation of abstract vector spaces. Quantum physics is a step further into other phenomena of the microcosm.

The vector is *state*, and the operation that changes it is *process*. However, processes are also vectors, so the quantum calculations become more intriguing. We tend to imagine particles spatially moving over time, separating their statics from dynamics, and sometimes notice that their arrangements and changes can follow similar patterns, but we might never have given it more importance if it weren't for algebras of quantum mechanics.

The electrical phase *photon* (particle of light, electromagnetic wave) is periodically replaced by magnetic. Electricity and magnetism alternately induce each other in standing waves *electrons* in the shells of atoms. Also the half-integer positive *spin* (internal magnetic moment) of a free electron is transformed into a half-integer negative by the emission and interaction of a (virtual) photon of a unit spin with another electron ($1/2 - 1 \to -1/2$) to then negative half transformed into positive by the following interaction ($-1/2 + 1 \to 1/2$), now a

[7]Isaac Newton (1643-1727), English mathematician.

[8]Albert Einstein (1879-1955), German-born theoretical physicist.

[9]Louis de Broglie (1892-1987), French physicist.

[10]Erwin Schrödinger (1887-1964), Austrian-Irish physicist.

virtual photon of opposite unit spin. This periodic time schedule of events corresponds to the wave structure of the electron.

The conditions remain periodic due to periodic processes and vice versa. We also see the effect of the process on the state in conservation the momentum, *inertia* of the body in motion. Just as action is transmitted through space over time, so it is transmitted through time through space. The body is where it is, because that was its most probable position, and then it will be directly there due to the inertia of probability[11]. More probable events are less informative, less effective and hence inertia.

The elementary particles of physics are in some of their changing states, but those many turn-turns remain what they were. The changes they undergo are limited in their goals, the need for the electron to remain an electron, for the laws of conservation to apply, and the like, as if they only fall into such troubles that do not essentially harm them. For particles that choose reusable processes, we say that processes according to states behave like states according to processes. A similar thing happens in particle fusions or decays when different states choose different processes.

The *dualism* of the linear operators and vectors on which these operators act guarantees us that the replacements of the ideas of space by time for physics will be non-contradictory. Besides, that is why we will find that this universe is composed of an equal number of temporal and spatial *dimensions*. Places and duration are equal concepts to isomorphism, mutually unambiguous mapping of corresponding concepts.

Quantum physics further tells us that achievement and development are not just two equivalents of phenomena, but each is also a reciprocal (regular, invertible) function. Globally, it allows the reversal of the flow of time, together with a change in the direction of the road and the charge. This does not seem to be in accordance with statistical mechanics and its irreversible breaking of the glass due to the one-way spontaneous growth of *entropy*, i.e. with the transfer of heat to a colder body. The paradox resolves the broader view, from the point of view of information minimalism. Let's look at how broader this new position is now on an example of economics.

Some company has production of items from three factories and delivery to three sales centers. Factories in a unit of time give 300, 300 and 100 items in a row, which they sell in centers at a rate of 200, 200 and 300 pieces. The cost of delivery per unit of an item from the first factory to the centers is 4, 3, 8, the second factory 7, 5, 9, and the third 4, 5, 5. The company has the issue of distribution with a minimum total cost. This is a typical task of the so-called *linear programming*.

There are several ways of delivery, even with the condition of the goods without the rest, but this is an example with only one minimal cost. I skip the solution (simplex method) that the first factory should send 200 and 100 pieces of goods to the first and second store, the second to the second and third 100 and 200, the third only to the third 100. If you are not interested in these numbers it is good again because the point is not there.

We imagine that the relationship between producers and consumers is a quantum process that translates the state of goods in production into the state of goods in sales. The condition of minimizing the cost corresponds to the principle of information and the principle of least action. The former earn as storekeepers or traders, and the latter as organizers of distribution, the first as representatives of the state, the second of processes.

Reversibility of (all) quantum operators would mean that for goods at the point of sale we can always know exactly in which factory they were produced. By the process which is

[11] The corresponding conservation law also applies to probability.

"always" in the "correct" state. The vectors are original (300,300,100) and copy (200,200,300). When such vector pairs are not equal, the quantum states and processes before and after can be significantly different. Anyway, they are still phases of a broader quantum process, here market economics.

When the original and the copy are equal vectors, there is no significant change in the quantum state (the atom remains an atom of the same type) as opposed to cases of fusion or decay of particles. Unless a different vector is formed, the constant of the proportionality of the image and the original expresses the *observable* that lasts, the probability of its observation which is therefore a real number.

The components of the vector and the coefficients in quantum calculus are also complex numbers. If they are not real numbers then micro phenomena cannot be observed. They are otherwise reluctant to declare themselves, as I said, because of the principled minimalism of communication.

http://izvor.ba/
March 6, 2020.

1.6 Selectivity

The informatics view of the world is broader than the philosophy of *materialism*. That is one part of this story. The rest are observations that matter itself is more subtle than the concepts we experience and that our perceptions only partially inform us about the environment.

Information is manifested by physical *action*, and it is the product of momentum and distance or energy and duration. Due to the law of conservation, both are discrete (finally divisible) and our observations are always final. However, physical reality is part of the infinite abstract world of truth and that is why information is selective and diverse.

When one of the two factors of action is greater, the other is less, and, in the limit case, all-spatial information becomes impulse-free, and all-time information is energy-free, like universal *theorems* that do not seem to belong to the material world. Each part of the theory constructed in this way is correct and agrees with every other part of that or any third correct theory (geometry, algebra, probability), although they can be divided into countless mutually independent *axioms* (Gödel's incompleteness theorems), selective and multiple.

More substantial truths are less absolute, in various ways. We know that the *universe* is expanding and that the limits of its visible part are further and further away. This phenomenon in "information theory" (mine) supposedly can arise from the accumulation of world history. By that assumption the space remembers, for the particle that travels the depths of the cosmos does not accumulate information about own travel. So space grows and matter is less and less. The same results come from the spontaneous growth of the entropy of the substance.

When we go down to even slightly lower levels of action, we find again a slightly different variety and "selectivity of information". For example, we communicate (interact), because we do not have everything we need, nor can we have everything, and then not everyone communicates with everyone.

Greater information comes from greater uncertainty, so uncertainty is proportional to the energy it transmits and duration. Hence, disinformation as well as anything that increases confusion can be energetic and offensive. Such a lie becomes part of a tactic, means or weapon in *games to win*. But not all games are games to win. The win-win game tactics in trade or the search for compromises in politics are examples of good games.

The opposite of peacekeeping would be the loss-loss tactic (lose-lose game), in chess gambit (sacrifice figure for position) or in business investing. Almost every game to win (winning game) wins the win-win game, and it is crucial to protect "goodies" from "evil", say for the stability of the economy.

Well, if "something" is being lied about, there's probably some kind of war of domination going on. The guests have "bad intentions" against the hosts and that is why they are lying. The home team, on the other hand, in front of their audience, accomplices or independents, also lie, and when the "match" starts, then some others lie, cheering for "theirs". And all this testifies about many faces of truth and its game of hiding.

Through the narrow windows of the senses we see just as much as we need. This minimum of communication for their own survival is required by all biological species, and these then converge around similar conditions, needs and conflicts. The principles of information and least action speak of such exclusion.

Stinginess is acquired, and from the surplus of physical actions, and then from the power of choice, living beings are created. Excess of the action, information or freedom, the nature solve when it can, spontaneously and selectively. There is always an environment around and if we choose, it is rather safety, indefatigably and purposefulness, primarily because of the thrift of the action, and secondarily perhaps because of psychology. Similar processes of reducing options, individuals and societies, are subject to the convergence of the senses of species.

There is no information without possibilities, and the more there are, the more choices there are or the less likely they are. With fewer chances, the information is higher, but the resistance to its broadcasts is also higher. Thus, the narrowing of perceptions and the growth of vitality are opposite aspirations, which tells us of our ignorance and the importance of the optimum. We believe that the world is simpler than it is due to the principle of information that we advocate for the need for security, efficiency or fatigue, and on the other hand we often mistakenly believe that we get more from more of everything. It is also selectivity.

At an even lower level, the peculiarity of snowflakes, tree leaves and all living beings in general, testify to how much nature resists abandoning its mixtures. *Law of Conservation* the amount of information intensifies this interference; it makes it even more difficult to reduce the number of options, to merge all forms of life into one species, one society, under one flag.

In a world of even smaller sizes, the Swedish physicist Johannes *Rydberg* (1888) discovered a formula for predicting the wavelength of a photon emitted by changing the energy level of an electron in *atom* using the principal *quantum number* (n). It is a natural number and one of the four quantum numbers that every electron in an atom has. As it grows in order, it denotes upper numbers of shells that are like concentric spheres farther and farther from the nucleus which bind the electron to the nucleus less and less, which, among other things, now reveals new types of information selectivity within atoms.

Information feeds on choices, but it is reluctant to consume that food. Materialist *philosophy* does not need this original diversity and it overlooks it, for it the universe is too large, the needs and scope of interactions are confusing to it. Communications, as well as living beings, come to materialist philosophy as surplus, and space, time and matter are inconsistent concepts. However, they are all tissue of the information and follow it's the same principles.

The foundation of the new philosophy is the smallest particles. As authentic representatives of physical information, its unpredictability, and they are so vague that it is not possible to know them exactly.

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March 13, 2020.

1.7 Dualism Lies

Science tacitly divides reality into a world of truth and a world of *lies*. The first of these worlds is divine, the second is devilish, some would say, and the question is whether we will ever be able to grab the "dishonorable" by the horns. That is the subject of this story. First, let's look at an example of my colleagues from Valjevo.

The citizens of city A always tell the truth, the citizens of city B always lie, and every citizen from city C alternately tells truth and lies. The firefighter on duty received a message on the phone from one of these cities: "We have a fire!" said one citizen. "Where?" The fireman on duty asked. "In city C" — the same citizen answered. Which city should the fire department go to?

The mentioned citizen could not be from city A, because they are constantly telling the truth, and the two statements received by the firefighter cannot be both true. He could have been from city B, citizens who just lie, because he could have lied "There was a fire with us!" And then again lied that he was "In city C", because the fire could have been in A! Citizen C, who would alternately tell the truth and the lie, could not say the first and then the second statement in that order. Also, it is not possible that the first statement was a lie (then the fire is not in C) and the second truth (the fire is in C). Therefore, the fire brigade should go to city A.

The lesson of this "task" is that the world of lies could in some way be consistent, similar to the world of truth. I will explain the anti-logic of that world through the algebra of the Irish mathematician George *Boole*[12] (1854) in which the only values of the variables are *true* and *false*. In applications, we represent them with the numbers 1 and 0 or the state "there is electricity" and "no electricity" in computer processors.

In Boolean algebra, each formula can be written using three operations: *negation*, *disjunction*, and *conjunction*. The first translates the value "true" into "false" and vice versa. The second and third include two statements each; disjunction gives them the value "false" only if both are false, and conjunction results in "true" only for both true. By translating these into circuit gates it is possible to simulate various logic processes.

On the other hand, there is symmetry of this logic. Every truth as a "reflection in the mirror" has some untruth and vice versa, so that the world of lies is exactly as consistent as the world of truth, in its twisted anti-way. For example, the conjunction "A and non-A" is always incorrect, it is a *contradiction* and in the world of lies it is the supreme value. If we copy it in the alleged mirror, the disjunction "non-A or A" turns out, which is *tautology*, the statement is always correct, the supreme value of the world of truth.

So, it is possible to look for total untruths in the very world of lies and then turn them into total truths by twisting (true to false and vice versa). But alas, just as it is difficult to understand a substance without space, it is difficult to find truths without lies or lies without truths. And that is the third part of this story, and it mostly concerns "information theory".

[12] George Boole (1815-1864), Anglo-Irish mathematician.

Math is a miracle. It is not the first time that she has been told that she is taking steps that no one will doubt in order to reach a claim that no one will believe. That's how we discovered that lies also inform us. What we could prove that cannot happen – it does not happen. That is why we consider "events" to be true (information is equivalent to action, and action is equivalent to happening). The opposite would be "non-events" (if any) from which it is not possible to obtain any information and which the true equivalents of untruths are. However, such cannot exist, at least not in a world in which every part of space, time and matter has some information.

In other words, the *universe* in which we live contains only the illusion of untruth. She is playing with us, because the truth likes to hide even though it cannot be hidden. The uncertainties that make up the tissue of information are inevitable, and nature does not seem to like them. It springs from the principle of information (which I advocate), its minimalism and the principle of least action (the latter is known to physics).

Nature does not have the possibility of classical lying, but it has incredibly great abilities to ignore and hide. Among us, the most famous of these are not declaring and reducing the chances of realization. On a scale of probabilities from zero (impossible event) to one (certain event), those more compact events of greater information are less likely and occur less frequently, those disjointed are more frequent.

Boundary, the densest would be impossible events, theoretical untruths, and the rarest would be theoretical truths. What we see is very likely somewhere in between. Due to the principle of information, nature almost never puts all its simple truths on the table, so that anyone who is too informed will become misinformed. Nature does not allow even the lies themselves, it over saturates and exposes the other extreme — the state of "completely uninformed".

If there are fewer chances for the event to happen, it is more informative and "hits" us more. At the other end of the probability are events with great causality, abstract, certain, poorly informative news and less material. They barely affect us, and we have no effect on them. What has mathematical accuracy is so intellectual and subtle as if it does not belong to this world; it is the opposite of what would have the ultimate inaccuracy, which would be excessively forcefully informative, brutal and destructive.

The principle of information thus initiates the principle of disinformation that just as the material world cannot do without at least some random events and therefore at least some information, it cannot do without at least some misinformation. By resisting the excess of "truth", nature has symmetrical reasons to resist the excess of "untruth". If you lie a lot, few will believe you, and if you only tell truths, they will find it difficult to understand you.

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1.8 Gibbs Paradox I

Just as politicians and historians regularly somewhat neglect the importance of inventors in the development of civilizations, and local masters and engineers neglect scientific discoveries, we all together underestimate the impact of mathematics.

So, too, Western social changes two centuries ago, due to the rise of liberalism and the industrial revolution, had their deeper root in the discoveries of probability, information, and quantum mechanics. Known as the French revolutionary, Lazare *Carnot* (1753-1823) was a mathematician and one of the first to (abstractly) explore the useful work of gases.

His son Sadi described (1824) the circular process of an ideal heat machine operating at the difference of temperatures of water vapor, heated air, or some third substance. The German mathematician Rudolf *Clausius* (1822–1888) gave it an even more precise form (1850) and used a later famous abbreviation, the quotient of heat and temperature, which he called entropy.

After two decades, the American scientist and mathematician Josiah Willard *Gibbs* (1839-1903) discovered statistical mechanics. He also discovered "information" which was then (1948) elaborated by Claude *Shannon* (1916-2001), an American mathematician. Gibbs, like many scholars from Clausius to this day, struggled to give the entropy some physical meaning, and in doing so, he was one of the more successful ones.

He noticed (or was close to) that entropy was growing by breaking glass and dissipating the pieces, which is why we call it "the clutter amount" today, and that the weakening of the bonds in the glass on that occasion speaks of energy and information loss.

This occurred in the shadow of *Napoleon*[13] Wars, the struggle of many peoples for independence, the decline of the Ottoman Empire (from 1299 to 1923), *Meiji*[14] Restoration in Japan, Colonization of Africa.

At the end of the 19th century, the Austrian physicist Ludwig *Boltzmann* (1844-1906) worked out the statistical mechanics of particle vibrations, atoms whose shaking gave heat and temperature, but he did not experience the victory of his ideas that came to life only after Einstein's explanations of Brownian motion (1905).

The falling gas temperature with the expansion of the vessel also bothered Gibbs when he (1875) made sense of the situation of his famous paradox of entropy. His thought experiment reveals the problem of mixing gas particles that we consider different in a situation where they are no longer. The resolution of the paradox is the (then absurd) treatment of particles of the same gas indistinguishably, such that when the permutation (swapping places) of two particles the state of the system does not change.

Why is it "paradoxical" to consider equal parts of a substance, what does "particle equality" mean and how far can we go with equalization? It is difficult to understand these dilemmas today, but I will try to explain their weight at the time by flipping a pair of the same coins. Each of the two coins has two sides, head or tail, so the outcome of the throw has four options: HH, HT, TH, TT. Over time, after many repetitions, the outcome of HH is about a quarter of all throws, so counting them we also know that we really had four equal options mentioned.

Let's apply this to particles of two types of ideal gases separated by a barrier inside an insulated vessel. Let them be two parts of equal volume, pressure and temperature. When the bulkhead is removed, the gases are mixed and spread throughout the vessel, each to twice the volume. The total entropy is higher, which will show both the calculus (omitted here) and the fact that restoring the bulkhead does not separate the gases back, that it is an irreversible process.

Then imagine that the same gas is contained in the both parts of the pan. By removing the barrier, there is no mixing of the "two gases" and spreading to twice the volume and the barrier is restored to its initial state. The process is reversible and the application of the same argumentation contradicts the account.

The point is that entropy does not notice the replacement of the place of the "same" particles, that such do exist, and that the number of combinations rather than variations is

[13]Napoleon Buonaparte (1769-1821), French military leader, born of the Revolution.

[14]Meiji Revolution, Reforms in Japan from 1868 to 1889.

then important for changing the energy and information of the vessel.

Therefore, entropy then is not an extensive quantity (proportional to the amount of substance), so if it depends on the possible distributions of the particles but not on the order of them, then dividing by the factor of the number of partitions (multiplying by a factor $1/N!$) In the calculation of the entropy will improve the calculations, and that really happened.

The solution of the Gibbs paradox indicates the subtlety of the two situations: the first particles in state A and the second particles in state B, and the first particles in state B and the second in A. Quantum states are vectors, compositions are tensor products, so the sum (AB + BA) is a symmetric state and the difference (AB - BA) is antisymmetric. Symmetric elementary particles are called bosons, antisymmetric fermions.

Experiments show that there are only two types of elementary particles mentioned, with no numbers other than plus and minus one between pairs of states. The same thing comes theoretically.

For example, by replacing two particles, the boson state does not change, and the fermion state changes sign. Twice this substitution restores the initial state, becoming the unit (neutral) operator, i.e. unitary, as are all operators of quantum mechanics. They are such that they would not change the (unit) norm of superposition of quantum states. The eigenvalues (of each) of these operators are only plus or minus one, so in no more complex system is there any mixing of bosons with fermions, since the superposition would give a third value.

The energy operator (Hamiltonian) is symmetric, so the potential and kinetic energies of the body are in boson-type fields, and therefore what the field acts on is fermionic. Photons and gravitons are bosons, electrons and protons are fermions. If the fermion state contained two of the same particles it would be zero (due to subtraction), and this cannot be normalized (into the unit intensity) and cannot represent superposition, so it makes no physical sense.

Therefore, two fermions cannot be in the same quantum state as the *Pauli Exclusion Principle* say, otherwise successful in explaining *Mendeleev*[15] tables of chemical elements.

It is difficult to understand how many discoveries in the chemical industry, energy, telecommunications are in the aftermath of this short story, until we look back at the way people lived in the 18th century. Even kings are not what they used to be, and all this is reversed by exact thought.

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1.9 Hamiltonian

Information is manifested by physical action, action is the product of energy and duration, and energies take two main forms, kinetic and potential.

The sum of the two is the energy of the body in motion, that is, the function we call the "Hamiltonian" after the Irish mathematician *Hamilton* (1833) who began building the physics on the law of conservation energy.

A gravity-free satellite is faster in a stronger field due to changes in potential energy, which is supplemented by motion energy. The sum of these two energies is constant until another force such as a rocket engine, cargo deflection, collision or friction with air acts on

[15] Dmitri Mendeleev (1834-1907), Russian chemist and inventor.

the satellite. Similar things happen to charged particles in the electromagnetic field, but also to other forces.

The states and processes of quantum mechanics are the representation of normed (unit) vectors and operators of abstract *Hilbert*[16] spaces, among which the *Hamiltonian* is one of the most important. It is the sum of the kinetic and potential energy operators and they are functions of the momentum and position operators. The actions of these over time are "realities" for further testing.

We know from classical mechanics that any change in total energy when a momentum changes will be reflected by some change in position over time. It says the first of the two famous equations of the Hamiltonians. The second says that any action that changes the (total) energy of a body by changing its position results in a reaction that changes the momentum over time. These two equations define the Hamiltonian and vice versa, the law of energy conservation gives these equations.

The elaboration of the above equations takes us into the amazing world of theoretical physics. The first step is the *Schrödinger equation* which, simply put, is the oneness of changing the wave function over time and the action of the Hamiltonians. We believe that all matter is in waves because the known physical phenomena of quantum physics are solutions of this equation, and we cannot prove by experiments those which are not its solutions.

The *commutator* of two operators A and B is the difference between their successive operations AB - BA. It is zero when processes A and B are independent, which according to the said Hamiltonian equations does not apply to momentum and position, nor to energy and time. Then these commutators are not zero but of the order of magnitude of the quantum of action, and their equations become *Heisenberg's uncertainty relations*. Physics is mathematically very connected, even when it speaks of indeterminacy over definiteness. This is why quantum entanglement is so confusing and *Bell's*[17] Theorem (1964) is incomprehensible to many.

This theorem proves a contradiction of the idea of hidden parameters that could supposedly avoid "spooky action at a distance" (Einstein) in quantum mechanics. The problem with it is the contradiction of randomness as well as consistency itself, where it is said that without uncertainty there is no certainty, without causality there is no coincidence! And that is hard to digest.

The wonders of the Hamiltonians are a story without end.

If in said commutator A is an arbitrary function and B is Hamiltonian, then commutator equals the change of function A over time. So there is no change of function if it and the Hamiltonian are independent phenomena. In other words, every change is about energy and time. The product of energy and time is action, and it is the equivalent of physical information, so information is the very essence of the nature of things.

From the above, it is easy to prove (algebraically) that the commutator of two momentums or two positions (different particles, places, and moments) disappears. By the way, the momentum and position commutator (the same particle) is not zero. These commutator relations are called canonical and are referred to in the most general form by Stone–von Neumann theorem (1931). We interpret it by reducing the mutually dependent phenomena (formally) to "position" and "momentum".

The question remains the understanding of the term "position" by "particle", because

[16]David Hilbert (1862-1943), German mathematician.
[17]John Stewart Bell (1928-1990), Northern Ireland physicist.

space, time and matter are only information (the hypothesis I hold), and each form of information is ultimately divisible, corpuscular. The solution to this problem has long existed in physics (Fock space, 1932), and it is not understood until the “theory of information”. And that, I hope, is some of the next topic.

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1.10 Bell Inequality

If we misunderstood a phenomenon, in experiments it might seem to us that the nature is deceiving us, and the deceiver would break up rendering the correct algebraic expression as incorrect.

This is the point of John Bell’s method of contradiction (1964), which challenged the idea of *hidden parameters* after the discovery of quantum entanglement. These hypothetical parameters and the “incompleteness of quantum mechanics” were an attempt to explain the *Einstein–Podolsky–Rosen paradox* (1935) in order to avoid “spooky action at a distance”.

Bell’s theorem and his further research proved such attempts to “rescue” quantum mechanics logically impossible. It is perhaps even more relevant to information theory because it supports the basic premise that – coincidence exists.

I tried hard and couldn’t understand it – is the usual statements made by students of theoretical physics. Bell’s theorem is simply too difficult to understand, but if it can be explained to laymen at all, then the American professor David Harrison’s example (1982) also helps.

We have three different properties A, B and C. The number of objects with property A but not with property B plus the number of objects with property B but not property C is greater than or equal to the number of objects having property A but not C. This is algebraic correct statement, check! Sum of numbers, number (A, not B) + number (B, not C) is greater than or equal to number (A, not C).

We call this relation *Bell’s Inequality*. Let’s test it in some (imaginary) room with different people. Let property A mean ’man’, property B ’height 175 or more’ in centimeters, property C ’blue eyes’. The inequality mentioned then says that the number of men below 175 plus the number of persons (male or female) is higher than 175 but who do not have blue eyes is equal to or greater than the number of men who do not have blue eyes.

As much as a room is, with as much as we want and with any kind of person, upper algebraic inequality is always true, it is indisputable. That is why, if we find the numbers in the room to be inaccurate, we will conclude that there was cheating, for example, some were entering or leaving the room while we were counting them.

The aforementioned relation is not thoughtless, it is a typical ending of calculating quantum entanglement and is similar to other Bell’s inequalities. Each of them breaks the idea of the assumed hidden parameters of the APR paradox – introduced to avoid “spooky action at a distance”. Measurement that would challenge Bell’s inequality would tell us that nature is cheating on us or that we are being misled in understanding nature.

For example, this inequality applies to stream of particles (photons, electrons). Spin is a measurable vector (by polarization or magnets). Let A be the orientation of the spin “up” (north), B the orientation “up-right” (north-east), C the orientation “right” (east). Then Bell’s inequality says: the number of electrons with spin “up” but not “up-right” plus the number with spin "up-right" but not "right" is greater than or equal to the number of electrons of spin "up" but not "right".

However, measuring the spin "up" and then "up-right" creates a paradox. Of all the electrons that pass the first filter, 85 percent of them pass the second, not 50 percent – which we would expect from a simple distribution (orientation of the direction of random vectors, spin, in the classical way).

The experiment shows that only 15 percent of electrons are spin-up with spin up-right, which corrupts Bell's inequality. In the example of counting people in a room, the analogy is that by determining the gender, we change the height of the person.

In fact, if this disorder of (expected) probability did not occur, it would mean that we had defeated *Heisenberg's uncertainty relations*. It would be apparent that the uncertainty in the quantum world is ostensible, that it comes from our not knowing of all causes, or that it has parameters that only need to be accounted for to achieve the universal causal reality. In contrast, in nature, some coincidences are objective and cannot be replaced by tricks.

From the point of view of the theory of *information perception*, I add, when we get some information from a coupled quantum system, we subtract some of its indeterminacy and so much determines it. It is synchronized, because other laws (information) that speak of certainty, non-coincidence also apply, and the way of this "strange" harmonization is confirmation of the informational nature of the physical world.

The fact that synchronization (of space, time and matter) can happen instantaneously and regardless of the distance of the parts of the entangled system, and possibly the action of the present on the past, more precisely because it confuses us, tells us that nature is not exactly what we think it is. On the other hand, we can think of dependent incident events as interesting "actions" created by a deficit of action.

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1.11 Pauli Principle

The Austrian-Swiss-American physicist Wolfgang *Pauli* (1900-1958) formulated in 1925 the principle of the exclusion of quantum mechanics, which, at the suggestion of Einstein in 1945, won the Nobel Prize in Physics.

This principle as a chemistry tool for building *Periodic Table* of elements is well known in its trivial form: there cannot be two identical electrons in the same atom. The real challenge is to explain it more accurately, popularly, without being banal.

Quantum mechanics is a representation of *Hilbert algebra*. Measurable quantities (observable) are coordinate axes, particles (quantum states) are vectors (probability distributions), and projections of vectors on coordinate axes give chances of measurement. In abstract algebra, a point is a vector, as is a "set of particles". Wave functions are also vectors, and the vector space is the solution of the wave equation. Such findings of math "spin in your head" and its point escapes momentarily.

We know that in reality, two different bodies cannot be in the same place at the same time. Similarly, when we throw two coins, each with a half chance of a head or a tail, we get four equal outcomes: hh, ht, th, tt. This means that there are always 'first' and 'second' coins, because otherwise ht and th would be the same outcome, with both hh and tt in about a third of outcomes in multiple throws.

Abstracting this concept, we look at two "particles" and two "states". Thus AB means "the first particle is in state A and the second is in state B", and BA means "the first particle is in state B and the second is in A". Then comes the hard part. When we say "two apples plus three apples are five apples" without considering the term "apples", only then are we in the field of mathematics.

Now try to abstract this, or at least move slightly towards the domain of "some" sizes, not caring about any. The states (particles) are "vectors". Their products are "tensors", vector compositions, and don't think too much about what those "sizes" might be. Bodies that cannot occupy the same space are not tolerant on each other, and their negation is the tolerant entities.

The sum of AB+BA would be one tolerant situation, because at the same time we have the first particle in state A (second in B) with the first particle in B (second in A). Such is *energy*, for example. It can be added to each other, kinetic to potential, no problem. So is space, so are opportunities before the outcome of a random event. This sum is of *symmetrical form*, because by replacing particles in states, it remains the same.

Then what would be the negation of this negation? This, we said, is an intolerable situation, but coming from its negation, we see that we must form it so that it cannot make physical sense if the first particle were the same as the second. In short, the negation of patience can only be the difference between AB-BA and nothing else, and here's why.

Physically realistic can be only such vectors that can be normalized (have unit intensities), which can always be achieved by dividing a vector by its intensity – except with zero vector. Only a normalized vector can represent the probability distribution, that is, the superposition of quantum states, and only one that cannot be normalized can express what we need now. Such is AB-BA, because considering the two given as the same particle, would give you zero.

Therefore, as tolerance is obtained through negation of intolerance, so the opposite of the situation AB+BA becomes AB-BA. The first states (sum) are symmetric, the second (difference) is said to be *antisymmetric*, and the first is called *bosons*, the second *fermions*. The former are named after Bose–Einstein statistics, according to the first of these authors, who was unhappily died at the time of the discovery, and the second according to Fermi–Dirac statistics. The bosonic probability distributions do not distinguish the individual particles, and the fermion particles differ, whether or not these differences are easily observed.

Mainly, I hope you understand. It would be too much for me to further explain why bosons have the integer *spin* (internal magnetic moment), and fermions are always halved, but from this one you also can feel part of Pauli's subtlety and ingenuity and Einstein's delight when he nominated Pauli for the Nobel Prize. .

The essence of Pauli's discovery was in the antisymmetric wave functions of pairs of points (particles, quantum states) and their projections to coordinate axes when measuring direction. The antisymmetric function by substituting the place of arguments (fermion) changes sign, so when two arguments are the same fermion, the negative and positive value of the function would be equal, which means that the mentioned function is null, there is none. It is Pauli's original principle that two fermions of the same species have a common wave function that is antisymmetric.

In other words, two fermions of the same species cannot be in the same quantum state simultaneously. This applies primarily to the four quantum numbers electrons in atom: the principal quantum number (n), the azimuth quantum number (ℓ), the magnetic quantum number (m), and the spin (s). In the same atom, two electrons, two protons, or neutrons cannot have all four quantum numbers equal, but a proton and a neutron can.

This idea was further developed by Russian physicist Vladimir *Fock* (1898-1974), and German Pascual *Jordan* (1902-1980) in a direction that in (my) theory of information will become particularly interesting. With Paul *Dirac* (1902-1984), an English theoretical physicist (1927) they laid the foundations for the so-called second quantization, formalism for describing many-body quantum systems, with multiple tensor product vectors, symmetric or antisymmetric, analogous to Pauli's. But that's another story.

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1.12 Globalization

One of the important points that information theory has so far mastered is the *principle of least action*.

It is not wrong to use this term, for information equal to that famous in theoretical physics from which we derive today all known equations of motion – from classical physics, thermodynamics, theory of relativity, to quantum mechanics. Physical substance possesses that laziness of physical action, and information in that sense is equivalent to action.

Substance tends to be inactive and less surprising, so the basic laws of physics are simpler and apply to simpler structures. Their information (action) will not just be delayed, but must be exchanged or the whole particle will be handed over. On the other hand, in trying to get rid of excess, matter becomes more complicated because of one mechanism that I have only recently known.

A quantum state is primarily one or more particles. We formally run it as a vector whose components give *probabilities* of outcome or – more physically speaking – it is a *superposition* of possibilities. More probable options are more often realized, but are less informative.

The conjunction of quantum states is the sum of the products of the corresponding components of two vectors (fidelity), so it also takes probability values (Schwarz inequality) and has the property that couplings of higher values are more often realized. Because they strive for less action, less information, which means more likelihood – quantum states associated!

Paradoxical situations arise. Going towards a smaller action, a physical substance spontaneously sinks into larger structures. It thus finds calm and new value. Like a disassembled machine that will not work, complex molecules have the properties at first glance absent in the atoms they consist of and may have a new quality that would be completely inexplicable without information theory.

Similar *synergies* (the interaction or cooperation of two or more organizations, substances, or other agents to produce a combined effect greater than the sum of their separate effects) exist in the higher levels of association all the way to living beings and beyond. Simpler life forms evolve into tissue cells of more complex forms, lower into higher hierarchies. The constant impulse to coupling mentioned the principle of minimalism, the tendency of the physical structure to get rid of its excess information (action) under conditions when all the surrounding substance is filled. That's how civilizations evolve too.

Driven by instincts produced by surplus of information the individual, among other things, strive for security and efficiency. By relinquishing personal liberties, we surrender them to the Pharaohs, emperors, kings, leaders, political parties, to increasingly "better" organizations, unaware of the deepest cause of universal subjugation.

Excess information means having more action, greater uncertainty and greater ability to choose. At a greater level of complexity, these structures cease to be subject to the study of physics; they no longer move through simple trajectories of the smallest action, such as the light bouncing off a mirror by crossing the shortest path between the two points, or as it would refract through the midpoints of different optical densities (speed) arriving in the shortest time, rather than becoming "disobedient". Then instincts, customs and legislation come into play.

What we consider to be the introduction of rules and order is actually the desire to get rid of surplus effects. It is useful to note that for a more intensive understanding of the function and evolution of society, the need for social discipline comes from the excess of information from individuals and the desire to reduce these surpluses. The individual is then freed from the unpleasant uncertainty, the need for urgent critical decision making, the risks, but so supposedly life lasts. We then favor the more capable king than the selfish pharaoh, governed by Rome over barbaric survival, the tranquility offered by churches and feuds of the Middle Ages.

Equality generates conflict, so it is also attractive as a good position to express our own excesses of vitality, but also to facilitate the creation of new hierarchies. The idea of equality through the rule of law was a full blow at the time of the French Revolution that its monarchy could not cope with.

But the vision of a (legal) system as an automatic machine that excels each individual in the ability to bring us safety and efficiency has the flaw (deficiency) as deep as invisible at the beginning. She stifles "aggression" by favoring her. By fostering equality, legal systems create an increasing need for legal regulation, and by not opening the door to new hierarchies they are digging its own grave, and digging it again by opening it.

Communism thus came to the point where it became merely a clamp and could not go on, and capitalism brought about corporations. It has brought us to a state of maturing new obsession with order and work, of companies and corporations, which for now is shyly recognized as globalism. The new motto is "Politicians are not the solution, but the cause of the problem!", and when these trends empower and ideas reach, things will turn compared to today and people will want globalism.

The history, of the clash of the rule of law and unworthy monarchs, threatens to be replicated in the clash of the idea of perfection of firm hierarchies against less effective and outdated politics methods. The initial surrender to the principles of law in monarchies resembles the corruption and manipulation of the political elite today. However, globalism will also have to offer security, meaning peace, prosperity and equality, much that it cannot achieve because of a deficiency, now of a different kind from the previous the legal one.

Corporations are essentially hierarchies designed to conflict with one another. They cannot cultivate sincere mercy on their subjects, because otherwise they will perish. They will feed on the need for money and their orderly society will be forced to evolve towards an ideal slave system. Thus, eternal opposites between desires and reality will grow, the same one hatreds that are actually the main feature of life.

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April 24, 2020.

1.13 Diversity

The essence of information is unpredictability. Repeated news is no longer original news and in this sense the very notion of multiplicity takes on new meanings. Until then neglected, the concept of diversity in (my) *information theory* is fundamental, and with the law of conservation it is particularly interesting.

The seemingly contradictory claims that this world is made up of news and news alone, that the news as soon as appear no longer are what they were, that their quantity (information of an arbitrary closed system) is constant, represent interesting discoveries. First of all, hence the *past* which is constantly accumulating and growing in a way that is never the same. Like everything else in this world, the past is a type of information, so it is also a type of action.

It is not possible to turn off the news so that nothing is left behind. As an elementary particle, news consists of options in shifting in a way that stores the amount of uncertainty given. Similar to a tour of a building, an object, an "something" when we see fewer and fewer of one facade to see more of other, the quantum of action is integrity, the smallest packet of information, and an elementary particle that, by more accurately observing the position, momentum becomes inaccurate.

In this way, information subtly connects us to the infinities that are the "normal thing" in mathematics. In this theory, infinity is the material of physical action, and its philosophy in this sense resembles *Plato's*[18] world of ideas (but there are and differences).

Aside from philosophy, we have the same in physics, for example in the movement of the water wave. What is in it moving about? The water particles go up and down and barely move left and right, and the wave goes perpendicular, forward. What is the substance, molecular, in the motion of a wave? If everything in the universe is made up of (parts of) a chemical substance, then the water wave is a paradoxical phenomenon. Here is one of the boundaries of the worlds bound by such a theory of information.

It is difficult to understand such a theory, even if it is impossible to hold that reality is only a concrete substance and nothing more. There is nothing better or more extreme about creating math. We discover it if its truths are pseudo-information (acting one-way), alias we are in this world both cause and consequence.

The assumption that information is ubiquitous implies the idea that space has memory needs to be checked. It would resolve the known paradox of the growing entropy of a substance on the one hand and, say, the reversible operators (evolution) of quantum mechanics.

Breaking the glass and scattering the pieces of the heart increases the *entropy* (clutter) of the glass irreversibly, and part of the (active) information of the substance becomes (passive) information of the space, its past from which the information (in principle) becomes more difficult to activate. Quantum mechanics operators, on the other hand, are all unitary. They are such that the normalized (unit) vectors of probability distribution (substitution) are transformed into the normalized ones, and the unitary processes are reversible, will preserve information.

Space thus remembers so the entropy of a substance spontaneously increases, which means that substances total energy and information are decreasing, in such way that the total energy and information of the universe remains constant. This very slow and persistent flow of substance melting and increasing space affects the expansion of the universe. As I have already written about it based on the principle of information, it is mostly one-way,

[18]Plato (429?–347 B.C.E.), Athenian citizen of high status.

because the nature is stingy with emissions of information, here from the uncertainty of space into the certainty of substance.

Additionally, that past may (must not) also act as the *dark matter* of the cosmos.

Due to the law of conservation, every aspect of information is finally divisible (because only the infinites could be mapped to their proper parts). More specifically, this refers to outcomes rather than opportunities, rather to current than potential information. Possibilities are boson forms (them may be more, they are tolerate), and outcomes are fermion (one by one, they are not tolerate), so the fermions are the ones, above all, which is the most *infinitely countable*.

When to almost every discrete outcome of a random event (fermion, vacuum site, moment of existence) is joined multiple options from which something might have arisen, we get an uncountable infinite set of possibilities. There are so many sets of states and then worlds that could (or would) happen, but didn't (or wouldn't) because the outcomes are unique (intolerant). The mathematics of infinity that came up with Cantor's set theory is not easy to retell. There are "equal numbers" of positive and all integers, because there is a bijection between them. Threading them from zero and alternatively one positive and negative integer, we can count all the integers, so its set is a "countable set" such as the set of positive integers.

The countable set is also a set of decimals of the number Pi ($\pi = 3.141592\dots$), but not the set of values that such decimals would give. All their values, the *continuum* of real numbers, are obtained when, in an infinite series of positions of digits, we vary infinitely them with decadal digits.

The uncountability of the continuum is proved by the contradiction of the assumption that there is an infinite series of all real numbers from zero to one. Namely, from the assumed sequence of numbers (decimal notation) we can see the first and the first digit behind the comma (let say it's digit a). We assign a different digit (x) to it. In the second number, notice the second digit (b) and join a different digit (y). Regarding the third digit (c) of the third number, attach to the new sequence the new digit ($z \neq c$), etc.

We lower new digits to a new decimal number ($0.xyz\dots$) that is real one, from 0 to 1, but not equal to any of the numbers in the given string! It differs from the first on the first decimal place, by the second on the second decimal, and differ from any in at least one decimal place; the new number is not in the given series which supposedly has "all" the real numbers. This is a contradiction with the assumption that real numbers from 0 to 1 can be sorted into one series. Therefore, a continuum cannot be equal to a countably infinite set.

From the infinite, therefore, there is the more infinite, and at the core of this story is that such abstractions are some information also. It doesn't communicate everything with everyone, so it might not be its end either.

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May 1, 2020.

1.14 Distributions

I work with *statistics* in social phenomena and I use probability distributions – a colleague asks me for advice – but in new situations I find it difficult to recognize the right one.

My best suggestion would be for him – looking at it more broadly, it doesn't matter, all *distributions* are the same, and in each of the situations it is possible to apply almost any of the statistics. But that proposal is not easy to understand. Without mathematizing, to analyze the nuances of seemingly very different distributions, to see similar broader patterns and their peculiarities at random under a microscope – is that possible?

The basic in probability theory is the *binomial distribution* (Bernoulli). The dice are rolled multiple times (eg 100) and the realizations of the same "desired" event are counted. When that one of the six numbers is, say, "five", the expected value is one sixth of all attempts (100/6). It is the average value and the most probable number of "fives" in the set of all (100) attempts. The chances of others decrease with moving away from that "expected" value. The mean deviation from the average is called the variance. In the binomial distribution, it is the product of the number of attempts and two probabilities, favorable and unfavorable outcome. The binomial distribution model has many applications.

For example, in the hospital, it was recorded that 75% of all patients (some diseases) die from it. What is the probability that out of five randomly selected patients, three will recover? Due to only two outcomes, it is a binomial distribution – the patient survives or not with probabilities of 0.25 and 0.75, respectively. In a series of five such three are favorable and two unfavorable outcomes, the cube and the square of the given probabilities, and their combination is ten. The product of the probability and combination for retail is 0.1. The chance of three patients out of five random recovering is 1:10.

When you recognize a binomial distribution in a given example, it will be easier in the next one. We count cars on a road and especially those that move at the "desired" speed. The number of favorable divided by the number of all, in a certain period of time, the probability is favorable. The added probability of unfavorable outcomes is one, so we have a binomial distribution. The distribution of oscillations of gas molecules is in the same scheme. The number of molecules is huge and the binomial distribution is approximated by the so-called *normal distribution*, exponential function of the square of the velocity. Its graph has the famous shape of a Gaussian bell with a vertex above the mean.

If we replace the velocity squares of the exponential function with kinetic energy (with additional constants), they become exponential functions of the probability distribution of the energy of the molecules. In the exponent, there are no (Gaussian) squares of variables, but first degrees, and the distribution is known as Maxwell-Boltzmann. Exponents are also probabilities, and their logarithms are then information. Visually completely different graphs are connected by internal logic!

Please note that in (my) IT interpretation, MB energy distributions would be *information perception* additions. They are individual "freedoms" that are each product of the "ability" of the subject and the corresponding objective "restriction" on a given (accidental) event. As the "information of perception" is close to the physical action, the product of the change of energy and the elapsed time, we formally come to the same, to the MB distribution.

From the examples of cars and molecules, we can see that the MB distribution requires air between its entities. This, in turn, indicates its "secret connection" with Barabási distributions of free networks, the arbitrariness of distance interpreted by that freedom.

Deeper in the microworld, there is no such commotion, the interactions of particles cannot be ignored and the need for a new formula arises. Then the division of elementary particles into fermions and bosons becomes important, the former with the Fermi-Dirac distribution and the latter with the Bose-Einstein distribution. As soon as we move away from the quantum world, the two distributions do not differ, the difference is so subtle. Both are equally well approximated by the thermodynamic (MB) distribution.

The reciprocal value of probability is a mean value of the "number of present" (equals) from which we randomly extract one. When we add one to that number of gas distributions (MB) we get the distribution of fermions (FD), and if we subtract one we get the distribution of bosons (BA).

These slight, quantum differences are easier to remember if we remember that fermions do not suffer in their environment like themselves, so let's say they present themselves as if there were more of them. Bosons are more tolerant and, conversely, pretend they are not cramped. The fermion distribution also works well in social phenomena in situations where there are several candidates and only one can be chosen. The bosons distribution, among other things, works better in the case of the mentioned free networks. How to understand it?

We transfer the law of conservation of energy or information approximately to the law of conservation of the amount of money in commodity-money transactions. When there are not too many individuals, the analogy with physics then goes further into its distribution. For example, if we freely add new ones with new links to existing nodes of a network, the chances of adding nodes with more links are higher. The degree law of *free networks* comes out of the account, with a very small number of nodes with a very large number of connections versus a very large number of nodes with a small number of connections.

Such are money flow networks (with very few very rich ones), internet networks (with rare concentrators), and even power lines of larger richer countries or social acquaintances and popularity, not only of people but also of movies, songs, politics. Such is the spread of infections in clusters, and recently the idea emerged that the same model applies to the spread of "accidental" murders in American centers and ghettos.

Because the *power law* applies to free networking it is an approximation of the boson distribution. Namely, the exponential function developed in series and the unit subtracted from it, becomes approximately a function of the power law distribution. With the fermion distribution, that unit would be added and the "magic" of transition to the power law would disappear.

You should now recognize the smallness of the fundamental difference of the leading distributions, but if you are looking for greater accuracy, this still popular story becomes something else. If it weren't for that, mathematical discoveries would be too easy and we would have been masters of the universe a long time ago.

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1.15 Fixed Point

As I will not hear about *Banach*[19] fixed-point theorem, so everyone knows that, these are the first words that children say when they speak — a colleague told me jokingly in a conversation about infinity in physics and continuing — I can hardly remember that from a study, remind me. I know that in mathematics, "there is more infinity than the infinite" — quoting me he asked further — but from your explanations, the transfer of the alleged infinity to physics is not quite clear concept?

I agree that infinity is a tough nut to crack for physics and that is why I am reticent in these texts, I say. Infinities are inevitable in mathematics because of the strength of the very principle of contradiction, and then they are part of (my) "information theory" because of the very existence of the idea of them. And, yet I am in no hurry, not for me but for others.

[19]Stefan Banach (1892-1945), Polish mathematician.

It is confirmed, for example, by the fantastic harmony of mathematical analysis. Although it was a stumbling block to Cantor's theory of sets and a reason for contemporaries to consider him frivolous, the idea of "more infinite than the infinite" prevailed by the power of its logic. *Gödel's*[20] Incompleteness Theorem (that there is no end to truths) is a turning point behind which the existence of infinity in mathematics can be taken for granted.

I believe that (my) information theory will join that development, and here is why. Pseudo-information (pseudo-reality) is the one that affects us, and we do not affect it. Such are, for example, mathematical theorems. How then to explain the *leakage of action* (information) from that pseudo-world towards us, from something that does not change or supplement? I think out loud and the interlocutor joins in with a question. The law of conservation doesn't apply to them? – He asked.

Yes, well done, that's one of the good options – I answered – Formally, it does not seem to be the only one, but for now we can consider it sufficient. There is no contradiction, because from the infinite set it is possible to separate out countable infinitely many of its elements and still leave infinitely many of them.

It opens up possibilities that are seemingly in line with many, even with the Gödel's theorem of impossibility. But it is understandable, I guess, that I still can't brag about it, otherwise many would say "I knew he wasn't normal" and the continuation of the theory would be even more anonymous. But let's get back to the topic.

For now, it is silent how, let's say, we have a continuum of points, some definition of distance and a mapping of space into itself. When it is so-called contraction, which means that the copies are closer to each other than the originals, then there is a single "fixed point", the point that is not moved by copying. This theorem was first precisely expressed and proved by the Polish mathematician Stefan Banach in 1922, and since then we have seen it everywhere.

It is the application of infinity. A simple example of the Banach's theorem is a terrain map placed on the ground of the environment it represents. Then there is a single point on the map that exactly matches the place on the ground.

We get the second example starting from an arbitrary triangle ABC. The midpoints of its sides, the points A' on BC, B' on CA and C' on AB form a new triangle A'B'C', and the midpoints of these sides new and so on. In the nth step, the middle of the page forms the nth triangle. It can be shown that the sequence of these triangles converges to one point which is the center of gravity of each triangle in the sequence.

A similar example is "picture in picture" (mise en abyme), placed copies of the picture inside the picture itself. A seemingly infinite series of recursions (procedures or functions that are defined by them) are obtained, which, according to Banach's theorem, contain one fixed point.

A little more "mathematical example" would be proof of the uniqueness of the solution of a "sufficiently regular" differential equation. In the space of functions supplied with metrics (defined distances), which is complete (includes limit values of arrays), with mapping of functions into functions that we then call the operator, the solution of the differential equation, if it exists, is a fixed point of that operator.

Schrödinger's equation is "sufficiently regular", as are almost all other imaginable wave differential equations, which means that each potential can be joined by a wave, but also vice versa that each wave can be joined by a potential. The first is well known in quantum mechanics, and the second is not: each wave can be accompanied by some information, its

[20]Kurt Gödel (1906-1978), Austro-Hungarian-born Austrian mathematician.

action, and then energy. The energy-frequency model of physics can then be applied to periodic phenomena in general.

The process of free fall in the gravitational field is an example of contraction mapping, which is especially interesting to us due to the principle of minimalism of information. The physical system spontaneously tends to a state of lesser information; in that sense, the satellite's orbit is its private minimum and "its fixed point." In the usual language of physics, the trajectory consists of the states of the smallest potentials of a given satellite, its potential energy which, multiplied by the constant units of its proper (own) time, gives its proper actions, that is, information again.

It is similar with all other fields of forces, because we also calculate the charge paths in them from the corresponding potentials without violating the rule that the lack of (relative) potential energy is attractive, and the excess is repulsive. All of this would make no sense if the use of infinity in physics were banned.

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1.16 Democracy

The lie of the naive seduces, and the cautious teaches. In that saying, there is a lot of sadness and joy of modern *democracies* and societies in general, similar in their desire to achieve some equality among people – whether they are obviously special or are in subtle nuances.

Just as equal starting positions of competitors promise better competitions, so do *equal opportunities* on the market through greater competition or various increasingly interesting conflicts that arise from modern principles of equality of persons before the law. We stand for equality in order to increase the possibilities, for the sake of progress, and for those who feel underestimated to follow us.

But, "no one has drunk a glass of honey yet, which has not convulsed it with a glass of bile," Njegos wrote. If something is good, it is not universally good. Better treatment today probably causes an unhealthy population tomorrow, state aid to a worse economy hinders the emergence of some better, employment of the politically eligible as much as it contributes to the rise of power through loyalty paves the way for its staggering due to incompetence. Promising competitions for the masses produce champions against whom the ordinary world no longer has a chance.

By squeezing the balloon, its contents overflow somewhere, the tension seeks its unfolding, and the effort for the equality of society flows into some inequality. These are just analogies, of course, but I hope they are useful, if not for understanding the proof of the impossibility of democracy that I will try to sketch, then at least for remembering the results.

On the path to communism, the tending to equality of the working class for each of the society gave birth to a lifelong president. Let us notice the same pattern in dictatorships in general because of their appeal to some equality without which the rule of the masses is not actually possible. Trivializing in the correction of "injustices" produced by the "good" modern legal system, in the race to regulate conflicts that it itself generates by insisting on equality, what we have recently recognized as "political correctness" in a negative sense is emerging. Giving priority to the "free market" resulted in the rule of wealthy individuals, bank owners or corporations.

It is unbelievable, but the principle of equality is the key generator of the mentioned inequalities. I have written about this absurd automatism from various positions – from listing historical events to its causes in the principles of information, and partly in the following way.

The mathematical model of *free networks* consists of nodes with equally probable connections. When we connect a new node with the old ones, the new link has the same chances as any other: that is why it is more likely to belong to a node with more links. It is a general form of equality. The speed of distribution of the *power law* then distinguishes a very small number of nodes with a very large number of links versus a very large number of nodes with very few links, so precisely because of the insistence on equality of connection!

The consequences of "free networking" are further treated formally, such as a multiplication table and often with a known intellectual effort to solve problems from practice using mathematics. Let's try to understand some general points first.

This well-known network legality also has its IT part. The number of links (denoted by x) of a node is proportional to the probability of connection (p), and since the logarithm of the probability (equally probable outcomes) is information, so the probability proportional to some degree of number of links usually denoted by "minus alpha" ($-\alpha$). Hence the "minus" that the probability p decreases when the number x increases, so the exponent α is always greater than one. In the most common situations, alpha is a real number between two and three.

The function of cumulative distribution is the probability that the number of links is greater than a given one (again x) and is calculated by adding (integrating) the previous probabilities. The result is the so-called *Pareto's law*, a type of power law with alpha by one less ($\alpha - 1$), between one and two, with a graph more similar to the line and simpler to estimate the number of events in a given period.

Pareto's law arranges acquaintances, quotes, books sold, receiving phone calls, likes, spreading the Internet, protein interactions, earthquake magnitudes, crater diameters. It is used in guessing the time until the next earthquake, flood, and asteroid fall.

If there were an infinite number of inhabitants on Earth, and the virus of an infection did not mutate, nor would the other conditions of its spread change, then the growth of the infected would stabilize over time along some (straight) line of Pareto's law. The stochastic phenomenon would then become causal. But in practice, there are limitations that can be included in the account, and hence the optimums, the extreme values of free network expansion.

The current achieved value divided by the optimal is the computable coefficient of the network, which is popularly called the *rule 80-20*. The "80-20" rule says that about 80 percent of free networks will be nodes with few connections, and the other 20 percent will be nodes with many connections. The interesting thing about this calculation is roughly this same relationship, its great independence from the type of free network.

Another interesting thing about free networks is that "my friend has more friends than me". There is more, and there is more charm in finding interesting things ourselves. I will add, free networks are also situations when one can lie freely, spread untruths relieved of the danger that someone will catch and punish us for that. For example, if this is the case with the placement of *fake news* on the Internet, then the "80-20" rule says that a lie spreads four times faster on the Internet than the truth. Approximately, this relationship could, I suppose, be the scope of written fiction versus learned works, the proportion of worse and best students in school or what an individual did not and did not understand in a lecture.

Finally, democratic rights include some rights to lie. One should be naive and not notice that people do not telling the truth and do not know how to read the truth about them, about the world and oneself from those lies, because then it is no longer matter of democracy, but it is up to the reader.

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1.17 Present

Where does the *present* come from? This is important, and perhaps the central question of my "information theory", to which the answer might seem so unusual that it is better to remain silent. This is part of that story.

In all physical interactions, something communicates with something, actions are exchanged and reality is defined. When one subject communicates with another, the two of them are mutually real, and if the other can communicate with the third, with whom the first does not have to directly, then we say that the third is also mutually real (some kind) with the first. The story of "reality" is thus reflected in networks and formalism, and the inconsistency of mathematics becomes the correctness of this unusual definition of reality based on chains of interactions.

Wherever there is energy and change, there are actions, and then information, but also vice versa, because information is omnipresent in this interpretation. In such conditions there is no place for an "abstract world", say logical truths that would be "out there somewhere" outside the physical and independent of it. Here, everything we observe changes us, even the discovery of Pythagoras' theorem, despite the fact that we do not change it.

An abstract idea that is timeless is energy-free (the quantum of action is the product of a change in energy and duration), but since we are not able to comprehend infinitely long duration, then we do not get even infinitesimal action. I am not claiming that infinities do not exist, by the way, but that what we can take from them is at most finite.

The known proof of *Noether's*[21] theorem[22] derived from Euler-Lagrange equations of motion does not apply to asymmetric interactions and therefore there is no law of conservation. This means that "closed" pseudo-systems can (do not have to) lose (gain) information and, in particular, it is possible to constantly subtract parts from the world of mathematical truths, while it remains equally infinite.

That there are more mathematical truths than elements of any conceivable set is guaranteed by Russell's paradox (there is no set of all sets), then proof of (Zermelo) set theory that there is a greater than every infinity, even Gödel's theorem of impossibility, and the world of truth is bigger.

In other words, there is the possibility of creating the present by filtering information from an infinite pseudo-world of truth. Through the sieve of the law of information maintenance and other principles of physics, abstract ideas are separated into forms of concrete reality. Along with taking over from infinity, and because real information "disappears" at the time when it "occurs", it also accumulates in the past. At the same time, unlike the unchanged amount of the present, for the future and the past, the law of conservation becomes debatable.

[21] Emmy Noether (1882-1935), German mathematician.
[22] see: https://www.academia.edu/39562358/Emmy_Noether_--_Poetry_of_logical_ideas

The world of the past is “pseudo” (false, one-way) because the past accumulates; new quantities of the present are constantly deposited in it and form a “false” reality different from abstract truths. Both are informative for (our) present and are able to act directly on it, but abstract truths do it subtly and almost always equally, and the past is less often the older it is. Like the effect of gravity decreasing with spatial distance, the influence of the past decreases with time.

The new theory of information occasionally turns out to be so shocking that it should be retold as a fairy tale, even if we believe that there is no fairy tale. But it is not enough to assume that the present awaits a certain, static schedule of events in the future, even if their choice is considered uncertain, in a theory that cannot do without objective coincidences. We would then make a mistake as in the interpretation of Heisenberg’s relations of uncertainty as a kind of only our ignorance, making then a theory contradictory to Bell’s theorem on quantum entanglement.

Due to the extremely low intensity of the actions we are talking about, the consequences of this aspect of the information theory are easier to see in cosmology. We know that the visible part of the universe is limited because its farther points move away from us faster and faster, so that those beyond the *event horizon* – which move away from us at the speed of light – escape us. That is why there is more and more space in relation to the substance within the universe visible to us.

I explained this expansion of the universe from the principle of minimalism of information and I would not repeat it now. In short, one of the explanations is, *space remembers* and matter melts into space at the rate of spontaneous entropy growth. What is more interesting now is the following hypothetical question: is the total amount (of information) of space and substance always exactly constant and how could we check this by observation?

For example, because substances are slowing down less and less, the expanding of the space slows down (never stops), unless the space receives additional information from the pseudo-world of truth. However, if the total arrival of the future into the present of the substance slows down, the time of the universe could also slow down.

Is it otherwise an unexpected consequence? It is not, because, for example, the relatively slower *flow of time* in the universe results from the increase in the “mass” (energy) of its space. Namely, the properties of the gravitational field may be initiated by the mass it contains at one point, but they are further maintained by the space that surrounds that mass. This turns out to be in line with Einstein’s theory of relativity, although it seems like a surprise.

In information theory, now all of a sudden, the meaning of a space that becomes increasingly, say, “thicker” to leave for later the term “pregnant”, and then its special contribution to relativistic effects (relative increase in energy or slowing down the time of the bodies in the field) becomes more visible.

http://izvor.ba/
May 29, 2020.

1.18 Quantity of Options

The screen image consists of a grid of *pixels* (dots). Older TVs and many 32-inch ones have a million (720p), smaller 49-inch models have just over two million, and newer 50-inch and more have eight million up to the latest with over 33 million pixels (8K). In order to see one pixel of such a TV, you need a magnifier.

The *screen resolution* refers to the number of pixels of the grid that makes up the image. Not to exaggerate, because that doesn't matter for the point of this story, I'll stick to the 720p resolution which means a matrix of 1280×720 pixels arranged in columns that make up the width of the screen and the rows that make up the height.

Each pixel can be controlled by the mains' voltage that defines its brightness and color. Light is digitized, said and quantized, usually in eight bits with which 256 (eighth degree of two) levels of intensity are achieved. Each brightness goes in three basic colors ($3 \times 8 = 24$ bits) which make a spectrum with more than 16 million brightness' and colors per pixel (256^3).

If we multiply the number of spectrum possibilities of an individual pixel by the number of them in the matrix, we get the amount of screen options, which is a simple number of screen possibilities. A more complex type of quantity of options is information. It is not the immediate number of possibilities, but the logarithm of that number. Information is also a matter of control.

We can arrange all rows of matrices into one line and consider pixels (spectra) as components of a vector. Dividing that string into two parts, and then the part that interests us more into two again, and so on, we get to one pixel. The number of divisions is the logarithm (base two) of the number of all and it is the screen resolution information. When the pixels have the same capabilities this is the *Hartley Information* named after the engineer of the Bell Company who first defined it in 1928.

One bit is one position with two possibilities, has or has not current, or 1 and 0, it is the basic unit of information. Three bits have eight possibilities (two to the third power, $2^3 = 8$), five bits 32 possibilities (two to the fifth, 2^5), and the product of the number of possibilities ($8 \times 32 = 256$) is the sum of the information of these possibilities ($3 + 5 = 8$). This is a feature of the logarithms that Hartley observed, enabling the Bell telephone company to start measuring the consumption of information analogous to the consumption of water or electricity.

The same applies to the collection of the *amount of uncertainty* of random events, which is then also called information. For example, there are 2 outcomes of tossing coins and 6 of throwing dice. Throwing both a coin and a dice has $12 = 2 \times 6$ outcomes, and their information is equal to the sum of the individual information of the coin and the dice. The logarithm of the number 12 is equal to the sum of the logarithms of 2 and 6.

The information of the pixel spectrum (8) multiplied by the current pixel intensity (number from 0 to 256) is the current activity of the pixels, and the sum of all these is the total intensity of the screen image at a given moment. When pixels are occasionally not active but are included by some distribution (independent event probabilities of unit sum), then the sum of probability products and corresponding information will give the average information of each pixel.

That average is *Shannon's definition* of information. It is named after the mathematician who founded it in 1948, also working in the Bell Company. A huge number of theoretical papers on the capacity of the channel that transmits information, on Markov chains, ergodic source, crypto codes and their applications, confirm the correctness of the ideas of Shannon and Hartley.

Note further that the variation of nuances in these definitions gives "surprisingly large" differences in the meaning of the terms. This is typical of mathematics and is a frequent cause of its alleged misunderstanding by laymen, and when it comes to new discoveries it is also the reason for misunderstandings among connoisseurs.

Information perception is the next step in information theory. It generalizes the previous two definitions, and it concerns physical action. For example, the emission of a screen image is also a matter of consumption or emission of energy over a given time. Multiplying unit information by energy and adding by spectrum and resolution we get the sum of the products of pairs of two sets of values, the sum of the products of the components of two vectors, which we call perception information.

The components of a vector represent individual dimensions of a vector space and their number can be huge. However, when the two such starts from the same origin point, they lie in a single plane, we say they span a parallelogram. The area that the *vectors span* is equal to the product of the intensity of the vectors and the sine of the angle between them. In addition, there are their mutual projections that are created by multiplying the intensity and cosine of the same angle. Both of these values are important in information perception theory, but they are not today's topic.

Common to the three definitions of information is their increment with the growth of the number of options and uncertainties. Outcome information decreases with probability, so the principle of more frequent realization of more probable events becomes the principle of less frequent realization of more informative ones.

Unlike the previous two definitions, there is no information of perception without energy or action (there is no change of energy without time). The mentioned, and otherwise new, principle of information minimalism is equivalent to the long-known principle of least action in physics. Then it is confirmed that the law of conservation applies to information (perception), such as the conservation of energy, and suddenly it turns out that such a law must also apply to probability itself.

I develop the theory of "information perception" quite lonely, but from years of work it is evident that it opens Pandora's Box not only of miracles within mathematics and physics, but from social phenomena, through biology, to artificial intelligence technologies. Hence, so many sequels to these more carefully viewed very different information stories.

http://izvor.ba/
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1.19 Flows of Events

Quantum physics divides *elementary particles* into bosons and fermions. The spin (internal magnetic moment) of the boson is represented by an integer, the spin of the fermion by half. The force fields are made up of bosons that exert their effects on the corresponding fermions. Bosons tolerate equals in the same place (space-time events), fermions do not. I'm just mentioning familiar traits.

Parts of electromagnetic radiation, photons to which *visible light* also belongs, are types of *bosons* — tolerant particles. They do not collide with each other, but are ignored, bypassed or interfered with, and the components of white light as elementary colors are obtained by refraction through Newton's prism. It then shows us that there is a quantum structure (in the photon of white light) locked in some process of coherence with its own law of conservation.

Unitary operators (processes) of quantum physics confirm this conclusion everywhere. By being *reversible* (invertible, regular) these linear operators that represent the quantum evolution, store and remember the information. Cases of ignoring bosons by bosons can be added to them as a kind of mutual *independence*. It is a disinterest similar to everyday life in tolerating something, someone's behavior, insults or praise that would provoke reactions milder than normal, that we would pay less attention to than usual, or would not touch us and be irrelevant to us.

The Pauli principle of exclusion applies to *fermions*: two identical ones cannot be in the same quantum state. For example, two identical electrons cannot be in the same atom. Electrons communicate using *virtual photons* that are always somewhere around, while the photons themselves do not have such an incident. It is a well-known process of *Feynman*[23] diagrams, and the overtone of his schemes that we now listen to is that a randomly taken type of particle communicates with a fermion rather than a boson.

Virtual photons constantly leave a given electron and, if one of them interacts with another electron, the photon becomes real and the two electrons are electrically repelled by the Coulomb force. As with boats on water, when we throw a bag of sand from one to the other, the subtracted photon momentum from the first electron is added to the second.

So it makes more sense to talk about photons without electrons than vice versa, about electrons without photons; as if there is a tendency to have more photons than electrons. With the square of the distance between the electrons, the probability of that interaction decreases, but its quantity is constant, so (recently) I assume that virtual photons propagate as concentric wave spheres, with the probability of action in smaller amplitudes stretching over the surface of the sphere. However, even that is not enough to explain where all these (virtual) photons come from and what if not all of them are realized.

In addition, it is known that in the mentioned exchange, for example, the spin of the first electron $+1/2$ decreases by the spin of the photon, then +1, and the second increases by that much, which means that the spin of the second electron had to be $-1/2$. The next exchange can only be reversed, to subtract the second electron spin from the unit and add it to the first. Hence, the following remark that the process of exchange of (virtual) photons between electrons is very selective. In this I see the confirmation of the "strange" new idea of "taking the present from the infinite" which befits the theory of universal information.

Even if you didn't think it was speculative before, you probably want to in continue. Holding that every event must go with a change of some action (read information) and vice versa, the knowledge of Pythagoras' theorem (or any other) should work on us.

However, the power of action of abstract truth depends on the objective amount of perception, and it is always some final value. Regardless of the fact that (if) the duration of the theorem is infinite and the energy that such a transfer could therefore be null and void, any physical perception is always finite. I emphasize once again, the product of energy and time is a physical action (information) with a positive minimum (quantum) whose one multiplier (time) if the other grows indefinitely (energy) decreases indefinitely.

The laws of conservation apply to physical actions as well as to energies, and these are therefore finite quantities. The finiteness of physical quantities can take over parts of the infinite in accordance with the laws of physics and its power of perception, while infinity remains unchanged, because its basic property is that it can be its real (proper) part. By subtracting (adding) to infinity a certain amount, it can remain the same, which is not possible with finite ones.

These well-known attitudes of mathematics still do not have their application in physics, and we are now connecting them to the world of elementary particles. To the above (hypo) thesis, that the "inflow of the present" (from the infinite) to bosons is less probable than to fermions, we further add the previously mentioned "melting of substance into space", i.e. fermions into bosons. The idea of such a course of events comes from the principle of

[23]Richard Feynman (1918-1988), American theoretical physicist.

minimalism of information, but it can also be derived from the (generalized) spontaneous growth of the entropy of matter. Anyway, the space of the universe is more and more, and the substances are less and less.

Because space becomes "thicker" with age and because it is the ultimate carrier of the gravitational field — relative time slows down. Due to the increase in the energy of space-time itself, the values of the energy tensor on the right side of Einstein's general field equations increase, so we can consider this phenomenon as a new-old effect of gravity. This is because gravitational attraction always leads to a slower flow of time, whereby the mentioned increase in space takes on the character of gravitational.

On the other hand, it is justified to say that time flows more slowly and because there are more and more bosons and less and fewer fermions. Interactions of the present with the infinite are less and less probable and transmissions into physical reality are becoming rarer, and I define the relative speed of time in information theory by the perceived "amount of events".

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1.20 Dichotomy

In religion, ethics, philosophy and psychology, *good and evil* are common *dichotomies* (division of the whole into two equal non-overlapping parts). Whether the development of information theory happens in the way I imagine it and it interferes with these, for now non-mathematical concepts or not — it is interesting to consider them. In addition, they are informatics more than solid phenomena.

The starting principle of information is its *comprehensiveness*. Honestly, several years ago, I imagined a "self-sufficient" universe of information only as an attempt to find contradictions, and it has remained so to this day. Reduction to contradiction is a powerful method of mathematics, typical of it, tested and reliable, but it often proves too difficult, and we cannot always count on it.

Deduction is also a method of mathematics, otherwise worthless when at the beginning of each chain "if is, then is" there is no proven truth. In that sense, it is a secondary way of final research (experiment is a kind of proof by contradiction) and that is why it is primary in manipulating the truth (for fun, marketing, in politics). This presumed exact beginning of the chain of implications in the topic of "good and evil" could be a universal possibility of mapping "correct" into "incorrect" and vice versa. This is evident in the operations of Boolean *algebra of logic*.

These operations agree well with the aforementioned principle of comprehensiveness, this one with the finding that information is action and it further with the understanding that action must be something that is true. Thus, we come to the conclusion that the world of truth joins the world of lies by *bijection*, a mapping that is both one-to-one (an injection) and onto (a surjection), to a completely new thesis with enormous implications. I emphasize, everything that can happen is true, but it does not have to be provable to us and, again, everything that is true does not have to be available to us.

The information theory I am talking about implies some coincidences, hence unpredictability, inability to communicate everything with everyone, and then the limitations of perceptions of any subject (living or inanimate being, body or particle of physics), so it is in itself in line with the stated attitude, that a lie is not what we thought it was.

It turns out that the seemingly foreign "world of lies" contains exactly as many truths, moreover, also the same ones that are in the "world of truths" known to us, packaged in ways that are harder for us to access. It is the first step, to see the universe of information with that inner symmetry, with the easiest way of knowing by reading immediate truths, and the others more tedious. The next step is to join the individual notion of "good" (one by one) of the logical value "true".

The idea of less and less good in order to eventually reach "evil" does not work now, because it would be reduced to polyvalent logic (true, perhaps, false) and the theorem there that any ambiguous logic can be derived from bivalent (true, false). Also, there is no longer a dichotomy of good and evil, and the question is what could this world be like?

When a huge meteor fell to Earth nearly 66 million years ago and killed the *dinosaurs* that dominated the planet for hundred of millions of years, the possibility was opened for mammals and eventually humans to develop and rule. Deforestation seems good to some, but it also deprives the environment of others and who knows what damage it does to this planet and then to us on it. By supporting the unsuccessful company with donations, the unknown self-sustainable is not allowed to exist.

In short, good and evil could be relative terms dependent on the observer. This view is in line with the information of perception, I note once again, because it values the perception of things always relatively, in relation to a given subject. We communicate because we do not have everything we want, nor can we ever have it, and because of the limitations of perceptions, the world is then always at least a little different for different subjects.

Because the essence of information is uncertainty, and *information is universal* (everything that exists has it and without it exists nothing), no subject can know everything; there is no standpoint, nor standpoints, from which the total universal and across-the-board truth could be derived. This would be a possible proof of *Gödel's incompleteness theorem* derived using the above information theory, or let's say a proof of the impossibility of the existence of a set of all sets (Russell's paradox), but, for now, it is only an evidence and another confirmation of this theory.

In that ancient saying that in every good there is something bad and in every evil there is good, for now we emphasize: to one good is to another bad. Something that is "bad" for all our previous civilizations may be "good" for some future, or for some other living species around us, but it doesn't have to be. Mappings between "good" and "bad" can be separated by cosmic distances or eons, but they are always parts of the information universe.

That such a model is logically possible; I will sketch the proof using graph theory. Let some points be given, nodes of the graph, with or without connections between pairs. An arrow (plus) can appears at the end of the link if the link for the given node is "good", otherwise it is "bad". If there is a link between two nodes, it can have two, one and no arrows. However, a graph is possible in which each node with a "good" link also has a "bad" link.

For example, nodes A, B, and C are assigned meanings in order: dinosaur survival, meteor destructive power, and mammalian development. Then it is A $\to$ B, B $\to$ C and C $\to$ A. First, we interpret that the greater destructive power of meteors has a worse effect on the survival of dinosaurs, then that the greater destructive power of meteors facilitates the development of mammals, and third, that the development of mammals begins at the end of dinosaur survival. Each of the three nodes is "good" for one and "bad" for the other.

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1.21 Fiction

Fictions in everyday speech mean the creation of a separate world through literature, film, painting or art in general, but also incorrect ideas that run through our heads. We believe that only living beings can have fiction, and that they generally experience it differently.

Although it is outside the materialistic models of the world, it is made up of some data that can move us and therefore it is part of the universe of information. Behind any information there is an action and vice versa — with any action comes some information, so fiction is the topic of "information physics". An important part of it understanding is reality through communication.

If two phenomena can communicate directly, we will say that they are mutually directly real, and if there is no direct communication, but there is a third phenomenon that could communicate with both, then the first two are (only) mutually real. Thus, photons are mutually real because they can communicate via electrons and are not directly mutually real because they do not communicate directly. In the world of information, all phenomena are real, not just primary or secondary ones.

The secondary includes the pseudo-reality with which we "communicate" one-way. It can affect us, but we not on it, like (I guess) a mathematical theorem or the past. There are also fictitious phenomena that can affect us, and we can affect them, but whose information is received differently by different subjects. God is partly pseudo-reality and fiction, and we have examples of the overlap of fiction and reality in the quantum world. I'll explain the latter.

If process A will not significantly change the state of x, then we have the characteristic equation $Ax = x$, otherwise we have some $Bx = y$. It is the alphabet of quantum mechanics, where A and B are unitary operators, and x and y are vectors. $ABx = Ay = z$ and $BAx = Bx = y$ apply to the compositions of these processes, so when $z - y = x$ we have Heisenberg uncertainty relations and, in general, if z is not y, we have non commutativity of processes A and B and reality overlaps with fiction.

Note that a similar formalism of quantum operators and vectors is applicable to the macro world of physics, with a clear phase separation, operations A and B. When there are no restrictions on the relations of uncertainty then there is no mixing of reality and fiction. And since each (unitary) operator can be decomposed into factors, then we have the complexity (composition) of the process up to their fragmentation to the bottom of the micro world.

In the quantum world, we do not consider living beings, but due to more dominant uncertainty, i.e. coincidence, the choice of micro world particles is proportionally larger. Where does this difference come from? First of all, it is a consequence of the law of large numbers, which increases the certainty of the "big world", and then there is the mentioned mixing of reality and fiction inherent in the "small world". The result is the absence of natural fiction in the world of the big and the absence of life in the world of the small.

The excess information created by synergy is a particularly interesting phenomenon. It occurs similarly to the excess action of storms that are born, scattered and end in a couple of hours, or can last and last like the "Big Red Spot" on Jupiter, defying principled minimalism. By skimping on information emissions, there is both an excess and a lack of vitality, just as the striving for as few effects as possible gives birth to a storm as well as calms it down.

The attraction of the information deficit keeps the electron in the atom from which it can erupt only by externally added excess (photon), but also such very selectively. Because information is equivalent to the product of energy and time (action), and units of time in the case of atoms can be unified, then we interpret the lack of information as negative potential.

Negative potential energy is more attractive than positive when it represents a lack of information, so then positive potential energy is repulsive. The deeper cause of this is, I repeat, the principle of minimalism of information, a very mild, ubiquitous and persistent force. On the other hand, diversity and pickiness are immanent to the world of information that defies the former and encourages storms and life.

Through the creation of surplus in the general aspiration to deficiency or the planting of untruths in escaping from lighter forms of truth, apparent dichotomies, fictions and various other expressions of the principles of information, nature's aspirations not to act while everything consists only of actions are incredible.

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1.22 Light

Light is a flickering vacuum. The number of vibrations per second, frequency f from 400 to 790 terahertz, makes information about the color of light: from red, through yellow, green, blue to purple. The light energy $E = hf$, and therefore the color, depends only on the number of oscillations. Planck's universal constant h is also the smallest physical *action*.

Frequencies outside the given (light) range belong to other photons, the smallest particles-waves of electromagnetic radiation. In fact, the product of energy E and duration t is equivalent to information, and the options that a photon could have can be reduced to some mean number N of equally probable outcomes, so that the logarithm of that number ($\ln N = kEt$, constant k) is just *photon information*.

The number N, the numerus of the logarithm, is an exponential function of the action, $N = \exp(kEt)$, and the reciprocal value is the probability of one of the equal outcomes, $P = \exp(-kEt)$. If the constant k belongs to a set of complex numbers, these probabilities take the known form of a wave function and, moreover, the solution is Schrödinger's equations for a free particle in general.

In this popular explanation, I intentionally do not avoid the mentioned "easy" formulas because of the demonstration of the simplicity of (my) information theory. By the way, this is one of the most difficult topics in the exact sciences, and Schrödinger's equation itself is one of the two epicenters of accuracy and the difficulty of intuitively understanding quantum physics. The continuation of the story is the wave nature of particles, interference and the problem of shape.

When the mentioned constant k is a complex number, the upper exponential and logarithmic functions become periodic and the energy transfer of the photon is a wave. Like a water wave that moves the energy of water horizontally, perpendicular to its molecules moving vertically, a photon is a vacuum wave without other very numerous vacuum options.

In official physics, *interference* is the appearance of the mutual influence of waves, the result of which can be their amplification, weakening or cancellation. This is considered a very complex physical process that occurs when waves interact in correlation or coherence, either because they come from the same source or because of (almost) the same frequency. Interference is known to all types of waves, light, radio, sound or, for example, water surface waves.

For us here, interference is the packing of multi-wave information. A simple wave travels horizontally, deviating correctly and periodically up and down from the main direction like a sinusoidal graph. A complex wave obtained by the interference of several of them would represent a curve whose parts also wave in practically innumerable ways. Sinusoidal disorders are again periodic and represent the “fingerprint” of the waves present.

Visible light interference will give *white color*. It can be decomposed into the elements by passing through Newton’s prism, which is, among other things, proof that interference does not destroy the structure of its components, and that there is no significant interaction of photons. Photons tolerate each other because they are a type of boson.

There are other interpretations of photon shapes. For example, photons as false balls were recently described by an Indian physicist (Narendra Swarup Agarwal, 2015). He showed that they too can leave a sinusoid-like trace and mimic the packing of information by interference. Here, this difference in the interpretation of shapes, on the other hand, we understand that shape is not important for photons.

We can also draw the last understanding from mathematics. Namely, almost any part of an arbitrary function can be used to construct with a given accuracy the periodic interval of almost every other function (Fourier transform, 1822). This makes it possible to represent “shapes” and photons in countless ways. In other words, I don’t consider it best to insist on a (unique) photon shape.

The principle of information minimalism is a special spice to the science of photons. It dictates that information arises when disappears, as if the information does not want to exist but cannot escape the law of conservation. Light and all other elementary information therefore simply said vibrate because they have a certain amount of data that they would be happy to resolve off if they could. Consistently further, information is pooled, synchronized and interfered and their freedoms are drowned in the group. For example, an electron, striving for minimalism, will join an atom and get rid of a photon.

With the new understanding, photons are like waves in a “sea” of vacuum. They are a surplus and a disorder that moves on the “surface” of a huge “mass” of space whose “interior” is a past that is constantly being deposited. Space is also information, so the universe is not equal to itself in any two moments, it is always news that it could exist in the “universe of information”.

According to such a (hypo) thesis, the particle-wave today could interfere with the corresponding wave that passed the same path yesterday. Confirmation of this strange conclusion can be sought in the timeless nature of some equations of quantum mechanics, for example, in calculating the interference of a photon with itself as it passes through two narrow slits.

Quantum mechanics uses this otherwise *Young*[24] experiment (1802) to prove the wave nature of light and to prove the wave nature of other particles, and the confusing interference of an isolated quantum (the smallest packet wave of probability) with itself for now has interpreted only by its splitting for the simultaneous passage through two openings and interference of parts after.

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1.23 Gravity

The basic “force” of information theory (which I represent) comes from the principled minimalism of communication, from the (hypo) thesis that nature prefers less emission of

[24]Thomas Young (1773-1829), English scientist.

information as it prefers to realize more probable outcomes of random events. We see that concept everywhere.

For example, it is in the capacity of news that the repeated fades or that too much information misinforms us, that in the multitude the data is hidden like a needle in a haystack. The tendency of knowledge to disguise itself, in its own way, is also found in the law of large numbers (LLN) of probability theory. The diminishing uncertainty of a larger mass is also a decrease in the freedom of movement of particles that are in the mass — due to the higher priority of the center of the *gravitational* force generated by them.

Let's abstract LLN from the gravitational field of large mass (particles of matter or energy) and again we will get more certainty. We will get a smaller inflow of the present and, according to the "information theory", a slower flow of time. Relative time flows at a speed defined by the amount of realized random events. From the general theory of relativity we have the same conclusion about the relative perception of time with the remark (I wrote in more detail earlier) that the deficit of relative time in relation to one's own (proper) is exactly equal to the excess of one's own presence in parallel-reality.

At the other end of the scale, in a quantum world of small magnitudes, uncertainties are just as dominant as a particle violating the laws of (large) physics; it is (temporarily) absent or is duplicated (simultaneously). Her multiple appearances were frequent enough that she could interfere with herself by going through a "double slot." Now I am talking about the famous Young's experiment (Young's double-slit experiment, 1802), which once proved the wave nature of light, and which confuses physicists today more than ever.

Photons (but also other particles) when directed individually towards a curtain with two close slits will pass as if they were interfering with themselves and on the screen behind they will form characteristic bands *diffraction*. This phenomenon, I believe (as I wrote earlier), was observed by *Everett*[25] (1957) and described as interference of copies of the same particle in *many worlds*. Because of similar "copies" that he considered to be made wherever they had at least a chance, he was so despised by the academic community that he left his work in science.

The commitment of an individual to the environment can be calculated by scalar multiplication of vectors that in quantum mechanics represent superposition, the probabilities of possible outcomes in the observable. The components of vectors give observational distributions of observations (Born's law, 1926) in relation to given circumstances (quantum system), so a scalar product that is never greater than the product of the intensity of the vectors themselves (Schwarz inequality, 1888) is not greater than one. Hence, the meaning of probability for these products.

Namely, the vectors of quantum states (particles) represent probability distributions, and they are therefore of unit norms (intensities). Their scalar products (the sum of the products of the corresponding pairs of components) are not greater than the products of their norms, of one, and have probability values. States unite if they have the opportunity to make the probability of their coupling higher, and they then gravitate into less informative.

The assumption is that a similar mechanism drives the evolution of life in general and makes the herd prone to subjugation or people to association.

There are various consequences of this, and one of them is the existence of borders. The accumulation of uncertainty increases the number of individuals, but reduces their significance and impact. The rise of some and the fall of others are found in an extreme we call the *optimum*.

[25] Hugh Everett III (1930-1982), American physicist.

Optimal is, for example, the state of a satellite in free fall in a gravitational field when its subjects do not feel external attraction. No matter what the gravitational field is around, there is no gravity inside the satellite at a given moment. It can be proved that this is the state of least action, because the satellites move along geodesic lines which are solutions of Euler-Lagrange equations. These are trajectories derived from the principle of the least action of physics, from which it is possible to derive Einstein's general field equations.

Information is equivalent to action, and in that sense, satellites move in orbit, adhering to the above-mentioned principle of minimalism of communication. The state of minimum communication is the state of minimum emission of information and both go with the most probable random events and the state of maximum entropy. All these phenomena are equivalent to each other and to the *principle of inertia* discovered by Galileo (1590), Newton (1728) and Einstein (1916).

Since I advocate the connection between spontaneous growth (generalized) *entropy* and the attractive force of gravity, some (well-meaning and others) correct me that the entropy of a stronger gravitational field must be higher, because for God's sake, they say, it is therefore attractive, and you ''made a mistake" into the opposite statement. I mention this as one persistent misunderstanding.

The body has the highest relative (own, proper) entropy in the state of free fall when the gas molecules in the room are evenly distributed. The lower is the entropy of the body standing below or above the orbit, because the molecules are then unevenly distributed, the lower ones are denser. So it would be according to *Boltzmann* statistical interpretation of entropy (1872), but also according to *Shannon* (1948) where an increase in entropy means a loss of information.

Thus, the entropy of a fixed point is lower in a stronger field and higher in a weaker one, and it is always lower than the entropy of a satellite in orbit (on a geodesic). That is why the satellite moves, or the field moves in relation to the satellite. In general, the greatest entropy of a body is in relative rest, when it has the least information, so it will not spontaneously pass into a state of motion, and hence the *law of inertia*.

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1.24 Many Worlds

According to the "theory of uncertainty", information and action are equivalent concepts, so are communication and interaction as well. All reality consists only of information, and this of uncertainty, which means that the realization of something is always possible, and everything is impossible. What follows are consequences that may seem so speculative to some that it is better to keep them quiet.

We say that primarily two subjects (particles, bodies, or people) that can communicate directly are real, and if there is a third with which both can communicate, then they are secondarily real. We set the relation *reality* so as to discuss its transitivity (if A is in relation to B and B is in relation to C then A is in relation to C). The goal is Everett's proposal *many worlds* of quantum mechanics and "information theory", and here's how.

Let's say we accept the explanation of the experiment *double slit* using the mentioned multiverse (existences of secondary realities that are not primary) and accept the "fantasy" that in the case of two places where a particle-wave can be found it remains in reality at one and goes to parallel (secondary) reality to another. If in the "other" reality it has the same choices, there is a chance that the same particle will reappear in the primary reality, now as a double. In *Everett's* time (1957) such ideas were insane.

Today, we know from experiments about the double appearances of the same particle-wave, and this explanation of its interference with itself is worth considering. Uncertainty is so dominant in the "small world" that the description mentioned there is more significant, that the notion of the real is relative, and the question now is can we somehow introduce and test such a description in the "big world" of physics?

Every body is a multitude of particles and its part always communicates with the parallel reality. To simplify things, we will talk about some mean value of the number of body particles in a given situation, by which we consider both the particles themselves and their positions and moment. Like, say, the average weight of a group of people whose value none of those present have, we can use (this imaginary average value) in an appropriate calculation.

Let the first body we observe be close, at a distance r, to some other much larger body of mass M, and we are distant relative observers. We say we are out of a two-body system. According to the above definition of reality, all particles of the first body are real with their proper (own), but they are not necessarily real with the relative. That part of the first body which is not part of the reality of the third, the deficit of the relative reality of the first body, is proportional to the total mass of the two bodies (approximately M). The deficit of the relative number of events that happen to the (first) body is proportional to that. Let me remind you, space, time and matter are information itself.

In *information theory*, the number of random events defines time. Let the proper elapsed time be t and let us denote the corresponding relatively observed one with t'. The relative time deficit is proportional to the total mass of the two bodies, $t' - t = t'kM$, where the coefficient k is a very small number that decreases with distance r. Hence, we calculate the dilation (deceleration) of relative time, $t' = t/(1 - kM)$. Such a written result can be compared with the one known from the general theory of relativity.

In *Einstein's* general field equations, $G_{ij} = T_{ij}$, on the left is the space-time geometry tensor, on the right the energy, and the indices i and j each take the values of three spatial and one time coordinates. *Schwarzschild*[26] solution of these equations applies to weak centrally symmetric gravitational fields such as the Moon, Earth or the Sun. It roughly (very accurately) coincides with Newton's gravity, but we usually represent it by means of the space-time interval expressed by the coefficients g_{ij}, the so-called *metric tensor*.

When we work with orthogonal (vertical) coordinates in a spherical system, of the $4 \times 4 = 16$ possible coefficients of the metric tensor, all 12 with different indices are zeros, and the remaining four define Einstein's *Pythagorean theorem*, i.e. the square of the length of the "diagonal" expressed by the sum of the squares of the "leg" 4-D space-time of gravity. In particular, the "time coefficient", g_{44}, which stands next to the square of the time coordinates of the mentioned interval, and which is greater than one, expresses the (slowed down) velocity of the body's time flow in the gravitational field.

It is interesting that this coefficient corresponds to the above result for the dilation of relative time predicted on the basis of Everett's idea of "many worlds", and that the slowing of time resulting from assumed parallel realities and (hypo) thesis that the flow of time is proportional to the number of random events agree also with the theory of relativity.

In this order of presentation, confirmation of the idea of parallel realities in Einstein's general equations, otherwise based on (Galileo's, Newton's, and then Einstein's) principle of inertia, was sought. But, once we accept the settings of "information theory", we will

[26]Karl Schwarzschild (1873-1916), German physicist.

derive Einstein's equations from them in the described way, then not necessarily mentioning inertia. The principle of inertia will remain a special case of the principle (minimalism) of information.

The space-time metric coefficient, g_{44}, otherwise indicates the relative flow of time. Einstein himself used to say that gravity pulls bodies towards a slower time, and now we add, because the lack of information is attractive.

http://izvor.ba/
July 17, 2020.

1.25 Space Memory

Space remembers. It is a warehouse of the past that we see with light and other particles from distant stars, in the distance of galaxies from which we read the age of the universe, in the deposits of "memory" in relation to which the water in the basin accelerates as it turns and spills, then perhaps in traces of memories we call *dark matter*.

Light year is the path that light travels in a year at a speed of about three hundred thousand kilometers per second. The nearest stars to Earth are three in the Alpha Centauri system, a little more than four light years away from us. They are closest to us, except for the Sun, which is almost 150 million kilometers away from the Earth, one hundred thousandth of a light year.

The diameter of the *galaxy* "Milky Way" is 100-180 thousand light years. There is our solar system with maybe up to 400 billion other stars. The Milky Way is part of a local group of 54 true and dwarf galaxies, a formation called the "Virgo Supercluster" with a diameter of more than 110 million light-years. It is estimated that there are at least 200 billion galaxies in the visible universe, and the most distant galaxy discovered to date is MACS0647-JD, about 13.3 billion light-years from Earth.

The distances between the galaxies, mega galaxies and quasars themselves are much larger than the interstellar ones; they range from several hundred thousand to several million light years. These distances are not constant and on average grow steadily over time, so we assume that the universe is constantly expanding, and then we calculate that its expansion began about 13.8 billion years ago with the "big bang".

Electromagnetic waves, which travel to us from the depths of space for thousands of years at the speed of light, are read by astronomers using the Doppler Effect and other laws of physics. They testify to the ancient places from which they started, with which science always has some new dilemmas. For example, the observed faster distance of distant galaxies could mean their faster movement in the past and the acceleration of the expansion of the universe, along with the slowing down of the present time.

From (my) "information theory" is the hypothesis that the observed expansion of space could be a consequence of *memory of space*. During the movement of particles, it communicates with space. Captured by the principle of conservation and minimalism, it lasts and tries to disappear, and because everything that happens is information, so is its duration and its history. As the elementary particle does not "grow", does not accumulate information in itself, the resulting surpluses remain along the way.

It is an alleged IT approach. Another way to approach this question is the mathematics of quantum mechanics. Hilbert vectors are quantum states, hence particles. But vectors (dual to the first) are also operators over them. This means that processes are "particles" (dual to the first) and as such are written in space-time. Processes are also subject to conservation laws that lead to what we now call space-time memory.

To make the matter more intriguing, there is also the theory of relativity, whose general Einstein's equations we can formally obtain equally by taking four of the six space-time coordinates of the universe. The three dimensions are then "spatial", and the fourth "temporal" – who's coordinate we multiply by an imaginary unit. We already have such a model with variables derived from three Pauli matrices of the second order (roots of a unit) and three quaternions (roots minus a unit). And that is the third way to conclude about the space that remembers.

The question of how "information theory" explains the appearance of excess information by the emergence of the history of particles seemingly out of nothing, given the law of conservation, is a possible answer in slowing down time. The units of time of today's present in relation to yesterday are longer, because there are fewer and fewer events, and there are fewer of them because there is less and less substance in relation to space. There is less and less information of the substance, because the entropy of the substance increases spontaneously, and with the increase of the entropy, the information decreases. I would not repeat the details of this now.

On average, galaxies farther and farther away from us are moving faster and faster, to those from the edge of the visible universe that should be fleeing at the speed of light. But gradually overcoming that speed is impossible, so we will constantly see through increasingly distant galaxies and a seemingly declining density of the universe. Compared to the past, our time is slowing down, units of length are decreasing and relative masses are growing. The present acts as if it is entering an increasingly gravitational field. This again connects the above three interpretations.

The same Einstein equations of the field speak about the effects of lowering the body into (real) stronger gravity, relative to the distant observer. They are written in 4-D space-time coordinates and refer to distant places as well as to the distant past. However, during t, the distance in the fourth coordinate is ct, where c is the speed of light in vacuum, and the square of that "distance" is a huge number, so the gravitational influences through "history" are negligible compared to those within the present.

Information theory suggests to us that information is two-dimensional. As when it comes from the distant present, it transmits force from the past to spheres *virtual bosons* whose surfaces increase with the square of the radius of the sphere, and with that surface the amplitudes and probabilities of transmission decrease, i.e. the chances of interaction with a given present event. Hence the decrease of force with the square of "distance", but, as we see, it is significantly faster in time than in space. At the same time, it is necessary to distinguish the action of gravitational virtual spheres that expand in more dimensions than, say, electromagnetic ones, due to which the field of the former is weaker.

A more significant decline in information transmission through history than through the present can also be explained by channel noise losses. Let that be the next topic, with the only remark here that different interpretations of the same reality are not foreign to mathematics. We know that there are more than 620 essentially different proofs of Pythagoras' theorem and that this does not mean that the methods, or the areas from which these proofs come, are in contradiction.

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1.26 Channel Noise

An important practical question of informatics that arises when designing or using a system for data transmission or processing is what is the capacity of a given system, i.e. how much information can it transmit at a given time?

Shannon's theorem (1948) says that *channel capacity* is $C = B\log(1+S/N)$, in bits when the logarithm of the base is two. This B (band) is the frequency range for signal transmission in hertz, S (signal) and N (noise) are average signal strength and additive white normal (Gaussian) *noise* in watts. The signal-to-noise ratio is usually given in decibels. The capacity is the highest upper limit of transmission.

For example, a typical telephone line with a signal-to-noise ratio of $S/N = 30$ dB and a band of $B = 3$ kHz has a (maximum) capacity slightly less than $C = 30$ kbps (kilobits per second). A satellite TV channel with a signal-to-noise ratio of 20 dB and a band of 10 MHz has a capacity of 66 Mbps (megabits per second).

The quotient of capacity and band (range) that reminds us of the quotient of weight and volume, which is why we can call it the specific weight of transmission information, is actually more similar to the quotient of heat and temperature and thus to *entropy*. A larger increase in the entropy of the thermodynamic system corresponds to a greater loss of information, so that C/B can be considered as a "loss" of transmitter information, or a "gain" of receiver information.

The entropy in the exponent, $\exp(C/B)$, is the number of some equally probable options. When we subtract the unit from that number, we get the number of possibilities of the Bose-Einstein distribution, $\exp(C/B) - 1 = S/N$. The reciprocal value of the number of possibilities, N/S, is the probability that describes the statistical behavior, here the boson, of one of two types of elementary particles characterized by the fact that at low temperatures they can be found in unlimited numbers in the same state of energy, in a phenomenon called *condensation*. From the mentioned Shannon's equation, therefore, we find that the ratio of noise and signal corresponds to the Bose-Einstein distribution.

I have described a simple calculation that could appear in physics textbooks (not yet), but what follows are deeper. We get that the noise is proportional to the probability of the boson. Hence, the first conclusion that less uncertainty of the place has a higher probability (finding) of bosons. Due to less information, they are therefore more attractive. The second conclusion will be that time passes more slowly in those places.

Both of these excerpts belong to "information theory". The first comes from its principled minimalism, and the second from the understanding that the present, that is, time, is created by the realization of random events. The first appeared because natural phenomena run away from information, and the second because natural phenomena consist only of information. We now find how bosons trace the probability of space.

We have the opportunity to check new attitudes with the general equations of relativity. These are Einstein's equations on the tensors of the 4-D geometry of space-time and energy. Initially, these energies are provided by mass, finally optional, which we consider to be the cause of the gravitational field. What turn out to be important now are the amounts of bosons that make space more likely and time slower. A higher concentration of bosons gives a denser field, we say a stronger gravitational force (a slower flow of time is gravitationally attractive).

In this sense, the agreement of Shannon's theorem on channel noise with Einstein's equations becomes unexpectedly simple and good when we notice that information can be potential (like six possibilities before rolling the dice) and actual (a single outcome after).

Bosons are generally analogous to potential information, but they themselves can be divided into virtual and real, again into (new types) of potential and current. Examples of these are photons (electromagnetic radiation) with which electrons communicate (see Feynman diagrams).

We will find different examples of the above attitudes in modern *cosmology*. The universe is expanding so that more and more galaxies are leaving us (on average) faster and faster. They accelerate towards the edge of the visible universe, the *event horizon* of the universe, the farthest sphere from us within which everything we can see is located, and which escapes from us at the speed of light. This process is conducted by melting the substance into space, while the present is stretched and diluted, constantly disappearing in the thickening sediments of the past.

Although galaxies are accelerating from us, they remain visible, because they do not reach the speed of light. This whole process is like observing from a safe distance a body falling into a black hole. It does not violate the laws of physics (say, conservation of energy, momentum, information), but its relative mass increases, units of length shorten in the direction of the center of gravity, time slows down to a stop.

No matter how much you look at that body, it does not reach the event horizon of the black hole, the sphere that surrounds it and, as far as we are concerned, the time on which it stands. The body gradually wraps itself around the sphere like a mantle, leaving only two-dimensional information.

From the point of view of the one who is failing, before he reaches the event horizon of the black hole, our time from the outside world seems faster and radial distances greater. For a body falling into a black hole, we are becoming more and more accelerated phenomena, as galaxies look to us.

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1.27 Graviton

Classical physics knows four basic forces in nature — strong and weak nuclear, electromagnetic and gravity.

For each of them, there are special particles that are carriers of the force field. They are *bosons*, one of two types of elementary particles characterized by integer spin (internal momentum), unlike the other type with half-spin on which these forces act and which are called fermions. The Higgs field and the boson named after him have been recognized recently (CERN, 2012).

Carriers of gravitational force are *gravitons*, electromagnetic photons (electromagnetic waves including light), weak forces — two types of W (Weak) bosons with opposite electric charges and neutral Z boson (Zero electric charge), and eight types gluon mediator of strong quark interaction. Unlike photons and gluons, weak force bosons have mass.

The existence of gluon has been experimentally confirmed since 1979 in Hamburg, Germany, and bosons W and Z a year before that, when weak and electromagnetic interactions were combined. We have known about photons for a long time, and gravitons have not been isolated in the laboratory until today.

The electromagnetic force between electrons and protons in a hydrogen atom is 10^{39} times greater than the gravitational force between them. That unimaginably large decimal number written with 39 zeros behind the unit is considered to be the reason for the weight of the graviton experimental proof. Despite that, we learn about gravitons from the known equations of gravity with great certainty. They travel at the speed of light and have no mass, or are close to it.

Quantum physics distinguishes virtual from real bosons. An addition to my theory would be that virtual photons propagate in the form of concentric spheres around an electric charge (electron), and not linearly, primarily because they are carriers of two-dimensional information, and then because of the area of the sphere growing with the square of the radius its amplitude and probability of interaction decrease, as does Coulomb's force. The wavelength of the virtual sphere does not change as well as any delivered momentum. Real photons travel in planes of polarization.

The basic assumption is that it is analogous to gravitons. The photon must have zero rest mass precisely because of the square decrease in force, which has been very exactly checked (Williams, Faller, Hill: Experimental Test of Coulomb's Law, 1971), which is why it must move at the speed of light. It is then a property and graviton of weak gravity (of the Sun), and let the question remain as to how long they are virtual. The spin of a photon is (plus-minus) one, the graviton is two.

It is complicated to explain why the graviton spin must be exactly two. In short, it comes from the metric, a symmetric 2-tensor of the gravitational field that is covariant, local, and tangent to the point of Minkowski space represents Poincaré group. Hence, mass 0 and helicity 2. Details on this can be found, for example, in the paper: Phys.Rev. 138 (1965), B988-B1002.

Given the law of conservation of total spin, especially graviton 2 and electrons $\frac{1}{2}$ and that they are elementary particles, it is unusual to observe that such two do not interact directly. If it exchanges spin at all, graviton acts on a multitude of particles as on a water wave, on the additional entity of the orthogonal motion of water to the vertical motion of its molecules.

As the additional information of the child on the swing in the park, on the synergy of the simple sum of the information of the child, the swing and the park, gravity acts on the abstracted excesses from the particles of matter. In that sense, gravitons are more subtle communicators than photons, and their number (within the visible universe) could be proportional to the total information of those parts of it. This result is essentially no different from the one recently obtained (Ioannis Haranas and Gkigkitzis, 2014) by calculating information according to the holographic principle.

Unlike electromagnetic fields, for which the law of conservation of energy applies, the energy leaks somewhere from gravity. The Soviet physicist Lev Landau was the first to say this publicly, and Einstein seemed to know that, because by parallel movement (translation) of a vector along a closed line of curved space, the initial and final position of the vector will have different directions, which indicates the presence of (gravitational) force and change of momentum, energy and now information.

In the "information theory", we place the gravitational field $3 + 1$ in $3 + 3$ dimensional space-time, so the mentioned "departure" of energy acquires a physical meaning. Gravity is therefore much weaker than electromagnetism, because it dissipates into additional space-time dimensions. String theory also predicts additional dimensions, but as microscopic volumes of cylindrical threads of visible space, while here we are talking about larger widths of time.

Unlike space threads, according to string theory, whose thicknesses we do not see because they are small, in "information theory" additional dimensions of time are large, but again we do not see them because the photons (with which we look) do not come from there. If we looked by gravitons instead by photons, we would also see those dimensions?

Graviton information is also two-dimensional and spreads on the surfaces of spheres, but in more dimensions than virtual photons. As one-dimensional large circles on the surface of a sphere that need to cover the whole sphere a lot, these two-dimensional spheres cover only pieces of 6-D space-time, and this coverage deficit speaks of the weaker force of gravity relative to Coulomb's force.

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1.28 Authority

Authority is the right to give orders, make decisions and extort obedience; it is also a person or organization that has political or administrative power and control.

Formally, in the perceptions of an individual, the authority appears as a series (values) of restrictions on appropriate possibilities, considering its ability to manipulate.

To every event that creates a situation, a problem for the subject, we attach some values of personal ability and objective limitations. An ordered set of abilities is called the *intelligence* of the subject, and the corresponding set of constraints is called the *hierarchy* of the environment. The product of a particular ability with a corresponding limitation is *freedom*. The sum of freedoms is "information of perception." With more general treatment, these terms are gaining wider applications.

When the sum of freedoms is an interpretation of the scalar product of vectors (arrays) of intelligence and hierarchy, we gain consistency. Compared to the *Ramsey*[27] theorem, which says that there is no absolute disorder (in a series of random words a meaningful sentence will sometimes appear, a preconceived notion will appear in the sky of clouds character), this product says that there is no zero hierarchy. With the previously assumed objectivity of some coincidences of "information theory", it further follows that each subject has some non-zero amount of options.

In conditions of (approximately) constant information of perception, a temporary limitation of external perceptions will result in an increase in internal (monk effect), and a limitation of internal perceptions by larger external ones (listening), while a permanent deficit of "hierarchy" will encourage an increase in "intelligence". Examples are (perhaps slightly) the greater intelligence of the older brother or the cognitive retardation of children as they grow up with intelligent aids.

Greater information of perception means greater *vitality*, stronger opposition to more difficult obstacles, such as Caesar's daring crossing of the Rubicon River despite a Senate ban, and less propensity to surrender to fate like a log dropped through the water. With more information (action) comes greater aggression, and with less passivity; the smallest have simple physical bodies to which the principle of least action applies.

The subject is all the more able to solve the situation if he makes the obstacle smaller, but that has little effect on the change of his overall freedom. Ultimately, *unlimited ability* would go with the absence of prohibitions and would not belong to the "universe of information", nothing would be unknown to such a subject.

Perception information is an inert quantity, as if it is an enemy to itself. With a deficit we are insufficiently informed, and with a surplus we become misinformed. Its optimums change

[27] Frank Ramsey (1903-1930), British mathematician.

slowly, so in a state of hopelessness we are more prone to inventing phantom observations and bowing to authority, and in conditions of self-confidence, others will find it harder to deceive us. Participants who have unsuccessfully solved tasks will find it easier to see characters that are not there, they will look for comfort somewhere, unlike those with restored self-confidence. That is why religion is stronger among the poor and disenfranchised, and its influence weakens with the growth of the authority of the rule of law.

An important property of the alleged product of arrays is Schwarz's Inequality: this intensity is less than or equal to the product of the "intensity" of individual arrays. These intensities are defined by vector theory consistently and very universally for different applications (to connoisseurs of algebra). They are crucial in quantum physics. Let's just say that a scalar product has probability values (a number from zero to one), if the vectors we multiply are probability distributions (outcome values of the complete set of separate outcomes).

For example, let's have two counterfeit coins with tails and heads falling probabilities of 0.4 and 0.6, and 0.3 and 0.7, respectively, with a scalar product of $0.4{\cdot}0.3{+}0.6{\cdot}0,7 = 0.54$. This product is larger when both sequences are increasing, which means better "alignment" of the two vectors, more geometrically speaking — their greater parallelism. This leads us to the assumption that in quantum mechanics, harmonized states (particles, vector representations) are easier to combine, because they make events more probable.

From there, the electron would descend to the lower shell of the atom releasing energy (photon) because the atom and the electron form a more probable coupling. They have a higher value of the scalar product of their superposition. To the well-known observation of physics that an electron will come out of an atom with the addition of energy, we now add that its smaller action (products of energy and time) within the atom comes from a lack of information.

The principle of minimalism of information is activating every time we work with physical potentials, because information corresponds to action. However, information is a concept broader than the four basic forces of physics, and we find the same mechanism, for example, in associating people into groups, or in the subject's attachment to authority. Authority is then a vector (environment) with which a given vector (subject) would be better aligned. For the sake of comparison, in the case of uncertainty of the momentum and the position of the particle, the authority is the position.

It is not wrong to say that authority is something like food, water or air to people, bad in both shortage and surplus. It is obvious that children love well-intentioned authority and that, in addition to that, for example, peer violence decreases, as well as crime within a disciplined army, despite the possession of weapons. In that interpretation, spoiled as well as abandoned children show signs of growing up with a lack of authority.

Equality generates conflicts in a way that competitions of athletes in fair conditions become fiercer, and the proclamation of certain types of equality (believers, castes, workers, and market conditions) encourages the dictatorships of the Inquisition, Napoleon, communist leaders, or the oligarchy of liberalism. It is known that distributions of equal probabilities have the maximum Shannon information, and now we only add that this leads to the aspiration of nature towards "authorities".

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1.29 Turning Point

We tilt the box to turn it for an angle, after which it rolls itself. *Turning point* is the crossing of the center of gravity of the box (intersection of large diagonals) over the verticals above the axis of rotation – the lower edge of the side on which the box falls.

We know that a full glass will fall before an empty one, because its center of gravity needs a smaller way to the turning point. Non-return point has and the raft on the river approaching the waterfall, the trigger on the rifle before firing the bullet, the differently adjusted chronometers attached to the board that can move freely on a horizontal surface will also spontaneously adjust and eventually synchronize.

The irreversible condition is sometimes the "movement of butterfly wings in Mexico that will cause a storm in Texas". I quote the "butterfly effect" of deterministic chaos theory founded by the American mathematician Edward Norton *Lorenz* (1917-2008) who said that a state of chaos is when the present determines the future, but the approximate present does not determine the approximate future.

Journalist Malcolm *Gladwell* wrote an interesting book, The Tipping Point (2000), in which he sought a critical moment, a trigger, or a boiling point, as he puts it, which triggers a "social epidemic." He described a famous event in American history on May 18, 1775, when Paul Revere and William Dawes decided to spread rumors about a possible attack by the English to the locals around Boston, moving in opposite directions.

The first was very successful in spreading the news, and the British attack on Lexington on May 19 met with organized and fierce resistance and suffered a heavy defeat. The other whistleblower failed. Analyzing this event, the author highlights three important characteristics of the person who initiates the "epidemic" of oral tradition.

He calls the linker, connector an exceptional individual who is somewhere around us, and we may not be aware of him, but who has a large number of acquaintances. Thanks to such, the messages in the (free) network between the two places arrive in five to six handovers. Connectors are people who know everything and everyone. In the theory of such networks, I add, they are a small number of nodes, concentrations, with many connections, as opposed to many other nodes with few of them.

Another important characteristic, according to the author, has a maven (connoisseur). So he calls an unusual person who in the given circumstances gathers essential knowledge and has information about various necessary things, likes to discuss it and be at the service of people. The third crucial feature is the people that Gladwell calls traders. These are people with special abilities to convince us of something when we are indecisive and distrustful. A situation that unites the mentioned traits (a link, a maven and a trader) in a person can make him the initiator of a "social epidemic".

Each of the three features is an image of *free networks*, formally speaking, named after the free, equal connections of its nodes. I explained how the equality of probabilities of connection in the construction of these networks leads to the separation of rare concentrators, such as people who acquire disproportionately large wealth on the free market, or rulers who stand out with greater power in conditions of equality. Their appearance is a consequence of the law of probability, or the principle (minimalism) of information, and if you will, of a nature that does not like equality, because its essence is diversity, the uniqueness of the individual.

In such model, when equality creates inequality by separating concentrators, the need for equality can be sensed in order to achieve uniqueness. Looking deeper, it is a generator that produces unique snowflakes, tree leaves, and people from formally equal laws. Hence, looking further, we will find the abstract universality of mathematics.

However, we are not going that far here. Equally fantastic is the accumulation of action (information) which, under the slight compulsion of the principle of minimalism, creates surpluses and life. And with that I will round out this story. If we had not witnessed the eruptions of geysers and volcanoes on planets pressed also by the mild and universal compulsion of gravity, we would find it hard to believe that the ubiquitous pursuit of less can produce more.

When we notice that the principled minimalism of information supports the separation of a living being from an inanimate one, a creature with greater possibilities of choice in conditions of striving for less, then we are not far from finding an important example of the mentioned dualism of equality and uniqueness. A living being goes through similar phases of youth, maturity and old age through which the storm created and guided by the "principle of least action", the well-known ubiquitous "force" of physics, passes.

The association of living beings, which we can also call living, goes through analogous phases. In the early phase, the individuals of such a society are uniform (stem cells) that over time specialize in different jobs in the service of the hierarchy. In this way, they hand over their own surpluses of "choice" (information) to the organization, in addition to trying to evolve into more efficient by sacrifices of life expectancy, intelligence or reproducibility.

The tendency to reduce information also exists in demonism, the worship of death, as well as in the desire for order and security. Other emotions are just smuggled in, clinging to this fundamental process of the principle of information using it, just as we use gravitational force in hydropower plants to get electricity.

Both, life and death occur at "turning points" and just as a woman cannot be half pregnant, so the dead do not return to the living. In short, this would be an introduction to an interesting plot.

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1.30 Delayed Gravity

When we define information as a quantity of options and as an inevitable part of any physical phenomenon, we get an interesting theory of information. In it, dark matter and dark energy, which cosmology recognizes today in the "errors" of galaxy rotation and in their inexplicable ever faster distancing, will be easily explained almost secondarily and as if they are uninteresting phenomena for science.

1.30.1 Communication

Space, time and matter consist only of information. This is the starting point of my *information theory.* It is accompanied by the laws of conservation of information [3], stinginess with it, i.e. its minimalism [2] and action – described in the books listed at the end and in this one.

In short, the information is the amount of data that lasts – otherwise we would have no experimental evidence. Although everything in nature consists of information, nature economizes with it. That is why it is easier for us to encode than to decode, it is easier for lies to spread than the truth. Bodies are attracted by striving for a more probable state, a state less informative. The information is all in the amount of uncertainty and that is why its initial values are consumed as soon as they are published. They exchange by interaction, because interaction is (also) communication.

Subjects (particles) therefore communicate because they do not have everything, and they cannot have everything because then they would not be objects of the *information universe*. Due to that, the space is constantly changing, and its changes are in "width" and "thickness". The opportunity to change space is the movement of an elementary particle whose duration forms its own biography, the history of an object that does not grow, but leaves its memories in the space through which it passes.

The growing thickness of space contains reminiscent of the substance that once moved through it, the memories of space that are also information and actions for the present. Increasing the influence of the past is exactly equal to reducing the information of the present, in accordance with the law of conservation and the principle of minimalism (economy of nature with information). The widths of the universe are obviously growing in front of astronomers' telescopes, but other phenomena related to these (widths and thicknesses) are difficult temptations for cosmologists, one as dark energy and the other as dark matter.

1.30.2 Entropy

Entropy (S) in Boltzmann's sense (1872) is the logarithm of the most probable, given the possible distribution of gas. It is easy to prove that this is a uniform distribution of molecules, such that the distances between them are approximately equal, according to the formula

$$S = k_B \ln W, \tag{1.1}$$

where $k_B = 1.38065 \times 10^{-23}$ J/K is the Boltzmann constant, and W is the number of actual microstates corresponding to the gas macrostates. In short, the Boltzmann formula shows the relationship between entropy and the number of ways in which atoms or molecules of a thermodynamic system can be arranged.

Therefore, an increase in entropy is equivalent to a decrease in information (Shannon, 1948), now say for the amount by which a uniformly distributed mass becomes impersonal, amorphous, like a soldier in a parade. Therefore, the spontaneous growth of entropy is a consequence of the principled saving of nature with information, and *generalized entropy* would refer to any spontaneous loss of information.

"Generalized entropy" is reduced to a substance, outside space itself. With such an additional interpretation, the spontaneous growth of the entropy of the universe becomes a "melting" of physical matter and an increase in space. The total sum of information about space, time and matter remains unchanged as the past increases – the decrease in the information of the present is compensated by the influx from the growing past.

In short, space expands the entropy of matter increases, its total information decreases because it is deposited in the past in (increasing) space from where it acts on the present in an amount exactly equal to the loss of information of the present.

1.30.3 Dimensions

Due to the assumed objective unpredictable nature of the options by which we define information, there are unrealized in the "information universe". Such a complex "present" becomes Everett's (1957) *many worlds* of quantum mechanics whose preservation of history requires three coordinate axes of time, given the three coordinate axes of space and the possibility of uncertainty along each of them.

I found various proofs of six-dimensional space-time within information theory. Along with the three known spatial dimensions (length, width and height), there are three temporal ones, or the even greater number of "temporal" dimensions that goes with more "spatial" supplied with coincidences. You can also find them in the mentioned books[28], or in my previous texts, so I skip that part. Notice that we are working here with additional temporal dimensions, unlike, say, *string theory* where we only talk about additional dimensions of space.

The part you should not skip is the matrix equation $\hat{\sigma}^2 = \hat{I}$ whose roots are *Pauli matrices*[29] $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$, in the same order:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

or $\hat{q}^2 = -\hat{I}$, whose roots are *quaternions* $\hat{q}_x$, $\hat{q}_y$ and $\hat{q}_z$, respectively:

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{1.3}$$

Three matrices $\hat{\sigma}$ and three matrices $\hat{q}$, here of the second order, can represent scalars in vector space, because the multiplication of scalars does not have to be commutative.

Namely, the *vector space* on the body Φ is called an additive commutative group X of elements x in which multiplication with elements from Φ is defined so that for each pair $x \in X$ and $\lambda \in \Phi$ there is $\lambda x \in X$. In this case, for all $\alpha, \beta \in \Phi$ and $x, y \in X$ is valid:

1. $\alpha(x + y) = \alpha x + \alpha y$,
2. $(\alpha + \beta)x = \alpha x + \beta x$,
3. $\alpha(\beta x) = (\alpha\beta)x$,
4. $1 \cdot x = x$.

We call elements of vector space *vectors*. The body Φ over which the vector space X is called a scalar body because its elements are called *scalars*.

This opens the possibility in the *Klein–Gordon equation* to select some other four of the six coordinates, some 4-D from 6-D space-time. Klein-Gordon equation with mass parameter m is

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0, \tag{1.4}$$

where approximately $c = 300\,000$ km/s is the speed of light, and $\hbar$ Planck's reduced constant. The complex evaluated function $\psi = \psi(\mathbf{x}, t)$ has the spatial variables $\mathbf{x} = (x_1, x_2, x_3)$ and the time variable t, and the Laplace operator $\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$ only acts on space variables. By substituting $x_4 = ict$, with $i^2 = -1$ and $\mu = mc/\hbar$, equation (1.4) becomes

$$(\partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2 - \mu^2)\psi = 0, \tag{1.5}$$

in natural units.

Einstein's equations of general relativity also have this symmetry of spatial and temporal coordinates, which is easy to check. Due to the understanding of time as an organization

[28]for example, 1.13 Space and Time, [2]
[29]see 2.4.6 Generalization

of past events, we will add an appropriate understanding of space for this symmetry. We also discover that the effect of gravity extends through all six dimensions of space-time, where the time coordinate contains a very large number (speed of light) whose square is an unimaginably large number, so the impression of gravity through the past is much muffled in relation to penetrating the present.

Gravity curves space-time making energy and information leak from the gravitational field[30]. Therefore, the gravitational action extends into 6-D space-time, unlike the electromagnetic one, which is limited to our real 4-D world. Accordingly, if we "looked" with gravitons instead of photons, we would "see" possible options, and not just realized ones. The consequence of this expansion of gravity on all six dimensions of space-time is its weaker force in relation to Coulomb.

1.30.4 Other effects

Due to the gravitational effect on the present, the deposits of the past form an increasingly stable reference system over time. We notice this in the spillage of water that rotates with washbowl, an effect that Newton (1687) considered as proof of the existence of "absolute stillness" or "absolute space", which Einstein later called the Mach principle (the dependence of a given mass on the whole mass of the universe).

A somewhat different proof of the gravitational activity of the past in the present is the movement of Mercury's perihelion in the direction of rotation around the Sun. Unlike Newton's theory of gravity, Einstein's predicts the movement of Mercury around the Sun not along the line of a static ellipse, but along one that slowly rotates with the planet around the Sun. As we know from general relativity, the angle of movement of the perihelion, α, expressed in radians per the revolution (the rotation of the planet around the Sun), comes approximately from the formula

$$\alpha = \frac{24\pi^3 L^2}{T^2 c^2 (1 - e^2)}, \tag{1.6}$$

where L is the minor axis of the ellipse, T is the period of the revolution, c is the speed of light, and e is the eccentricity of the ellipse. Due to the influence of gravity from the past, we also expect that the angle of change of the perihelion decreases with the square "time distance", $X_4 = iTc$, so that this confirmation of general relativity can be considered as a confirmation of its addition presented here.

The influence of *dark matter* on galaxy rotation could be another piece of evidence, for example, if it turns out to follow the motion of masses.

The explanation of the *dark energy* by the theory presented here is a little more complex. The space of the present becomes "denser" not only because of the increasing size caused by the "melting" of the substance, but also because of the growing "deposits" in the past. The present, "observed" from some fixed moment of its past, would act as a body falling into a gravitational field.

Relative time would flow more slowly, and radial lengths would become shorter. The relative mass and energy of the body of the present would increase so that both the relative and the proper (its own) laws of conservation remain valid.

However, viewed from the present, movements in the past would be faster. The relative accelerated flow of time of galaxies that we observe further and further in the past due to

[30] see 1.13 Space and Time in [2], and 1.27 Graviton here

the distance from the Earth is compensated by a relative slowdown due to their movement. This also applies to mass, energy, as well as length. We see similarly in the movement of satellites around the Earth, where time flows faster due to altitude, but slower due to speed. These two values do not have to be completely undone.

Finally, notice that by traveling through space it is not possible to reach its edge, the *event horizon* of the universe, because due to the increase in the space of the present it moves away at the elusive speed of light. Analogously, it is not possible to go back to the beginning, to the *big bang*, because in an imaginary journey through the past we would need more and more time to infinity, so in this theory, in that sense, in fact, there is no beginning of the creation of the universe.

1.30.5 Epilogue

From this short article you can see how unusual the "information theory" I am working on is. That is the reason why I do not try to convince anyone of its possible accuracy, at least not until I am sure of it myself. In any case, I would be grateful to the reader who would discover and point out to me the inconsistency of this "theory", so that I would not bother with all that.

`www.academia.edu/`
August 25, 2020.

1.31 Neighbourhood

We see that information is a matter of perception in the differences in the perception of one's own (proper) and relative observers of the body (system) in movement or in the ways of communication (interaction) of objects, and even in relativizing the severity of the problem depending on the solver's ability.

The part of *infinity* we receive is always finite and subject-dependent in a way that opens up new questions about the sharing of information. We consume all-time truths in limited portions on the basis of which we assume their uniformity and universality, and then, due to the properties of the action of information, the energies of whole truths are null and void as opposed to the pieces we take. That seemingly absurd view of information theory is the subject of this story.

The famous saying that "what we cannot explain we do not understand" (Einstein's), which suggests the need to explore ambiguities, reverse and notice that with the understanding of infinity in our hands we are already standing on the threshold of the world of physics with ready explanations. We will understand the insensitive phenomena step by step as physical and in the future, similar to the accepted atoms and quanta.

This is not particularly unacceptable. Truths concerning discrete, countable infinite (discrete) sets are mathematically indisputable, moreover they are more accurate than physical ones, and intuition is what hesitates and hinders us. It will give way, and just as we missed infinity in the very abstract statement "$2 + 2 = 4$", our animal heritage will eventually digest future steps.

Namely, while the abstract "two plus two is four" we abstract from "two apples plus two apples are four apples", then "two legs plus two legs are four legs", then "two kilograms of iron plus two kilograms of iron are four kilograms of iron" and so on, by constantly listing the concrete, we realize that in the final many steps it is not possible to prove that one *abstract statement.* It is a miracle of mathematics in the fabric of concrete science. Even in one simple and abstract expression, the whole infinity of our reality is.

Abstractions are an important tissue of physics, whether I understand it or not. Without *complex number*, fantastic and practical, there is no exact science. Their infinities are so woven into the "concrete" that we no longer notice the extent of mathematizing the very concepts of, say, a "real" number or line; today we barely understand our ancient consciousness. We accept new practice unaware of new knowledge.

Nearby are points of complex plane. It is determined by two vertical straight lines, a real abscissa (horizontal axis of numbers) and an imaginary ordinate (vertical real axis), we would say even more abstract than the complex numbers themselves — if not for the coordination of airplane flights on London Heathrow (or another larger airport). Elegant, fast and accurate ways of marking, directing and tracking airplanes there are performed using complex plane theorems.

In construction, we can work with a modest knowledge of geometric planes and believe that there is no infinity, but if we try to make a *building of geometry* on this concept, another truth will appear. Just because we can't eat them, smell them, grab them with our hands, or fit them into mystical explanations of the world known to us from ancient times, we don't consider complex numbers (recently discovered) unworthy. Consistently, any future knowledge could be unworthy, because originality always defies some old beliefs.

To the mentioned sum $(2+2=4)$ the statements that the naturals $\{1,2,3,\dots\}$, the integers $\{\dots,-1,0,1,2,\dots\}$ and the rational numbers (fractions with integer denominator and numerator) has an infinite number of, and there are more irrational ones (real numbers that are not rational). The former are countable, we say discreetly many, and the latter are uncountable, the continuum many.

Records of rational numbers are periodic, irrational ones are not. In decimals $x = 0.232323\dots$ the pair "23" is infinitely repeated, so $100x = 23 + x$, and the number $x = 23/99$ is rational. All periodically written numbers are so rational. Non-periodic, such as pi ($\pi = 3.14159\dots$), are irrational. There are *discrete many* of the former and *continuum many* of the others. These are two infinities of different sizes.

The unpredictability of the next digit raises the order of infinity similar to the randomness of particles. All the ever realized states of all the elements of the universe are a discrete set, but their possibilities form a continuum. This second, the superset, is so many times bigger than the first that there is no chance that we will randomly choose the element of the smaller in the bigger one, even if we choose countless infinitely many times!

We are smaller than a drop in the ocean of possibilities, but our reality is "thick everywhere". There is an expression for a set of rational numbers (fractions) on a number line where it is also *everywhere dense.* No matter how small the neighborhood (positive length) of any rational number (point) of real axes, there is always another such number (rational, besides many irrational ones). So, logically, a well-arranged world is possible as a set of rational numbers, not nearly as big as a continuum, but "dense everywhere" in it.

The ultimate divisibility of physical information makes the space-time event of the 4-D world discrete, everywhere dense in a set of possibilities, and although the smaller never leaves its domain to the rest of the larger, without the larger it cannot exist.

We say that the interval of real numbers is "closed" if it contains boundary points, and "open" if it does not contain them. The ball is *closed* or *open* depending on whether it contains an outer sphere or not. Complementary to the open is a closed set (if a point belongs to one it does not belong to another) and separately each final set is closed. Most gatherings are neither open nor closed, and empty and entire spaces are the only open and closed sets at the same time.

Physical information is a discrete set and therefore closed, so mathematics teaches us in this case that there is no empty *void*, nor *universe* that contains everything. We must treat the vacuum and the whole universe (and only them) both at the same time as real and as pseudo-real information! The intersection of any collection of closed sets and union of finally many such are closed sets, which brings us back to the previous one — that there is no way out of real information from its world.

`http://izvor.ba/`
September 4, 2020.

1.32 Adherence

When we have an idea about something, then there is information about it, and with information comes uncertainty. The first follows from the assumption that we live in the *universe of information*, and the second that with the knowledge we get something unknown. Abstract performances with such (hypo) theses literally become the subject of "information theory".

Mathematics of numbers that does not target quantities has long been developed in functional analysis, topology, and set theory. The network of their views is a good model for further clarifications of the idea of uncertainty, and we will add it to the observation (previous columns) that empty and all space as sets that are both open and closed can be considered a door between *infinity* and *finite*. Infinity "leaks" towards us filtered by the laws of physics, and we will see why such an interpretation is necessary.

First of all, it is said that the laws of conservation (matter, energy, momentum, information) follow from the finality of phenomena. Only infinity can be its rightful (proper) part and be constantly spent and always remain the same. Additionally, topology teaches us that infinite sets (besides closed ones) can be the only open ones, so they can constantly be subtracted and last usable, among other things, because the union, no matter how many open sets, and intersections of finally them, form the open sets.

All its models are almost equally good for us, and if mathematical analysis seems difficult for you, it is often enough to imagine only "intervals" of numbers, such as (1,2) in which real numbers are greater than one and less than two. The same rules as the common axioms of both, model and application, give common consequences.

A point on a number axis is "interior" of an interval if the interval is its environment[31]. The collection of all interior points of a given set forms the *interior* of the set. Obviously, the interior of a set is its subset, and the interior of a set of rational numbers is an empty set. That is why we define an "adherent" point of a set in which each neighborhood has at least one point of that set. The collection of adherent points is *adherence*. Each set is a subset of its adherence. The adherence of an open interval is a closed interval; the adherence of a set of rational numbers (fractions) is a set of real numbers.

The interior and adherent points of the sets and their complements are mutually exclusive. Hence, the need to define a point on the *boundary* (edge, border) which is at the same times an adherent point of both the set and its complement. The boundary of each set is a closed set. These are just the first notions of analysis and topology, otherwise non-trivial

[31]A point $s \in S$ is called interior point of S if there exists a neighborhood of s completely contained in S. The set of all interior points of S is called the interior, denoted by $\mathrm{int}(S)$.

(hard) areas of mathematics. It has been said many times that such starting points are from attitudes that are difficult to doubt in order to reach not only the places we did not hope for, but also those that are not easy to believe at first. That's why we're not in a hurry.

The formalism of mathematics is the basis, but both the base and the superstructure are in the universe of information. What would previously be "hard to believe" becomes the relationship between the outcome of random events and all possibilities, the connection between reality and parallel realities, or 4-D and 6-D space-time.

The most countably infinite (discrete) set consists of the events of one reality which is the present (3-D space at a given moment), our reality of all particles of the universe, but of that size is and 4-D space-time developed layer by layer following one present. With various flows of the time, similar events (particles) reach all possible pseudo realities.

It can be shown that there is an isomorphism (mutual unambiguous, one-to-one mapping of structures) between the relations of these events and the relations of rational with real numbers. The adherence of a set of rational numbers is a set of real numbers. The universe of one reality is discrete (as rational numbers) in contrast to the continuum (size of real numbers) of the universe of all possibilities, and the second (larger) is the adherence of the first (smaller). It is useful to know for further work.

Here we will note that the union of information is also information and that the described 6-D space-time also contains uncertainty. It is information and therefore possesses uncertainty and exists in uncertainty. In other words, 6-D space-time is not the end of the story. Knowing *Russell's paradox* (there is no set of all sets) or *Gödel's* incompleteness theorems, the new interpretation of reality does not surprise us, it gained in importance. However, it opens a new perspective on the physical understanding of the present and time in general.

Let us consider this unusualness of the 6-D space-time model together with the interchangeability of three spatial and one temporal (*ict* - products of imaginary unit, speed of light and time) coordinates with some other four (out of six). I am talking again about the symmetry of space and time, which is directly verifiable in *Klein–Gordon equation* of quantum mechanics, but also in Einstein's general relativity, and the specificity is information theory.

We also discover it in the unpredictability of both time and space, visible from the limited speed of the subject's movement and the dose of the unknown in the movement of particles. Because we define the course of time here by the amount of random events, and then because of *Bell's Theorem* (1963), according to which we cannot outsmart Heisenberg's relations by introducing hidden parameters, we need a little more uncertainty than that possibly deposited in a static 6-D event pool.

In order to avoid circumventing the "phantom action at a distance", the uncertainties of time should be reinforced by the random emergence of the present from infinity. The set of possible events would then not be like a container of fixed options from which random outcomes would jump out, and it would not be possible to "deceive" the relations of uncertainty or to challenge Bell's Theorem.

The space-time of Everett's *many worlds* (1957) is no longer a set of given points, i.e. space-time events, on which the *present* moves in a random way, but the possibilities are further blurred by the inflow from infinity filtered by the laws of physics.

`http://izvor.ba/`
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Glava 2

Formalism

This is the third book in a trilogy about the world of information. In the section on the formalism of the first, Physical Information [3], I have tried to change Shannon's definition as little as possible to obtain information to which the law of conservation applies, more precisely to show that such a formulation is possible. The initial reason was to convince (some) colleagues who would say that it was okay for me to stand for "physical information", in the sense that "information" in reality may be subject to some kind of conservation law similar to that for energy, momentum or spin, but that story "is not a possible" as far as mathematical formulas are concerned.

In the second book, Minimalism of Information [2], the section on formalism is devoted to the analysis of the weightless state of a body falling freely in a gravitational field so that the reader may take a step towards generalizing the situation in terms of "principles of minimalism" from the side of nature. A free-falling physical system, such as a satellite in orbit around the Earth, moves along geodesic lines that are solutions to Einstein's general equations.

It is shown there that these geodesics are also solutions of Euler-Lagrange equations of motion according to the principle of least action. Christoffel symbols (necessary for defining these equations in the tensor form) were also derived from the same principle, and finally Einstein's general equations were derived from them. Thus, the idea of inertial motion and weightlessness is reduced to the principle of least action.

What is more important in that book is the connection of the "principle of least action" known in physics to the new and unknown "principle of the least communication". The body falls freely so that it communicates minimally. It then has the greatest entropy. It also follows a (relatively) most likely path. As it does not feel the force of gravity then, it does not feel the "spontaneous attraction of entropy" towards the center of gravity. In other words, if we consider that the spontaneous growth of entropy is the cause of gravitational attraction, then the largest entropy of their own have bodies in geodesics.

In the book before you, the Action of Information, the chapter on formalism opens up the application of infinity in physics. If I remember the difficulty with the "ordinary" (discrete) infinity in mathematics at the time of the discovery of functional analysis (limit values, derivatives and integrals), and then those "unusual" (continuum and larger) infinities of Cantor's set theory, no I hope for an easy understanding of my idea that abstract mathematical truths are also types of information, and even less about the "leakage of information" from that abstract world into the physically concrete. That is why here is a shy, barely noticeable introduction of physical action in infinity.

2.1 Vector space

The same two properties apply to addition and multiplication of real functions as to addition or multiplication of numbers, the operation "+" or "·" which is indicated by a circle here:

$$a \circ b = b \circ a \quad \text{(commutativity)} \tag{2.1}$$

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \text{(associativity)} \tag{2.2}$$

Thus, the written operation "circle" is easier to generalize on binary operations between vectors, matrices, and even on the compositions of functions and the like. Then it is easier to talk about the structure of the set $\mathcal{G}$ supplied with the given operation. When the existence of a "neutral" and "inverse" element is added to the above two properties:

$$(\exists e \in \mathcal{G})\ a \circ e = e \circ a \quad \text{(neutral element)} \tag{2.3}$$

$$(\forall a \in \mathcal{G})(\exists a^{-1} \in \mathcal{G})\ a \circ a^{-1} = a^{-1} \circ a = e \quad \text{(inverse element)} \tag{2.4}$$

we get the structure of a commutative or Abel *group*.

When we say a group only, we mean on a structure with an operation that is not commutative, is without (2.1). Such is, for example, the multiplication of matrices or the composition of functions. We easily find that then

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1}, \tag{2.5}$$

because, say:

$$(a \circ b)^{-1} \circ (a \circ b) = (b^{-1} \circ a^{-1}) \circ (a \circ b) = (b^{-1} \circ (a^{-1} \circ a)) \circ b = (b^{-1} \circ e) \circ b = b^{-1} \circ b = e.$$

We check that a group can have only one unit element as follows:

$$e = e \circ e' = e',$$

where e and e' are assumed to be two unit elements.

Ring $\mathcal{R}$ is called an additive (with operation "+") Abel group $\mathcal{G}$ with at least two different elements in which each ordered pair of elements $a, b \in \mathcal{G}$ are associated with a single element $a \cdot b \in \mathcal{G}$ which we call the product of the elements a and b. The multiplication in $\mathcal{R}$ has the following properties:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{(association)} \tag{2.6}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{(left distribution)} \tag{2.7}$$

$$(b + c) \cdot a = b \cdot a + c \cdot a \quad \text{(right distribution)} \tag{2.8}$$

A ring is called commutative if $a \cdot b = b \cdot a$ for each pair $a, b \in \mathcal{R}$.

Body Φ we call a ring with the property that the set of all non-zero different elements from Φ forms a group with respect to multiplication. We denote the unit element of this group by 1. The commutative body is called the *field*[1]. Examples of bodies (fields) are rational, real and complex numbers with common addition and multiplication.

Vector space on the body Φ we call the additive commutative group $X = \{x_1, x_2, \dots\}$ in which multiplication with elements from Φ is defined, so that for each pair $x \in X$ and $\lambda \in \Phi$ is defined $\lambda x \in X$.

In this case, for all $\alpha, \beta \in \Phi$ and $x, y \in X$ is valid:

[1]Definitions taken from the reading [16].

(i) $\alpha(x+y) = \alpha x + \alpha y$,

(ii) $(\alpha+\beta)x = \alpha x + \beta x$,

(iii) $\alpha(\beta x) = (\alpha\beta)x$,

(iv) $1 \cdot x = x$.

The elements of vector space we call *vectors*. The body Φ over which the vector space X is is called the scalar body because its elements are called *scalars*.

Examples of vectors are *oriented length*. They are equal when they have the same direction (are parallel), orientation (aim) and length (intensity), so the class of equals has its representative, oriented along with the beginning at the origin and the top with a given sequence of coordinates. The sum of the oriented length given by the sides of a parallelogram from the same vertex is the diagonal of the parallelogram from that vertex. The difference of these vectors is another diagonal oriented towards the vector from which is the subtraction.

Arranged *strings* that could represent the coordinates of vertices of the oriented lengths are also an example of vectors. They are added by adding the k-th ($k = 1, 2, \dots$) component to k-th, and therefore multiplied by the λ scalar by multiplying each component of the array by that scalar.

Quantum states are vectors. Their components are mostly complex numbers and such that they express the probabilities of observables (measurable physical quantities). We call them superposition, but also probability distributions of observables, which mean that quantum mechanics observes only their representatives of unit length.

2.2 Unitary Space

The vector space X over the body of the scalar Φ (real or complex numbers) is called *unitary* if each given pair $x, y \in X$ is associated with one and only one scalar $\langle x|y\rangle \in \Phi$ so that:

1. $\langle x_1 + x_2|y\rangle = \langle x_1|y\rangle + \langle x_2|y\rangle, \quad x_1, x_2, y \in X$;
2. $\langle \lambda x|y\rangle = \lambda\langle x|y\rangle, \quad x, y \in X,\ \lambda \in \Phi$;
3. $\langle x|y\rangle = \langle y|x\rangle^*, \quad x, y \in X$;
4. $\langle x|x\rangle \ge 0, \quad x \in X$;
5. $\langle x|x\rangle = 0 \iff x = 0, \quad x \in X$;

The function defined by $\langle x|y\rangle$ is called the scalar or internal product of the vectors x and y. The real unitary space ($\Phi = \mathbb{R}$) is often called the Euclidean vector space, then $x^* = x$. A complex unitary space ($\Phi = \mathbb{C}$) is in use in quantum mechanics; then $x^* \in \mathbb{C}$ is a conjugate complex number of $x \in \mathbb{C}$.

Note that it follows from conditions 1 and 2

$$\langle \lambda_1 x_1 + \lambda_2 x_2|y\rangle = \lambda_1\langle x_1|y\rangle + \lambda_2\langle x_2|y\rangle, \tag{2.9}$$

which means that the scalar product is linear *functional*[2] by the first argument, and from condition 3 it follows:

$$\langle x|\lambda_1 y_1 + \lambda_2 y_2\rangle = \langle \lambda_1 y_1 + \lambda_2 y_2|x\rangle^* = (\lambda_1\langle y_1|x\rangle)^* + (\lambda_2\langle y_2|x\rangle)^* = \lambda_1^*\langle x|y_1\rangle + \lambda_2^*\langle x|y_2\rangle, \quad (2.10)$$

which means that the scalar product is an antilinear functional in the second argument.

Unitary space does not have to be of final dimensions. In the unitary space, just like in Euclidean spaces, the notion of orthogonality and the orthonormal system of vectors can be introduced, and in the final-dimensional case, the existence of an orthonormal base can be proved.

2.2.1 Information of Perception

Let Ω be a space of random events and let $\omega_1, \omega_2, \dots, \omega_n \in \Omega$ be experiments about which some *individual* (living or non-living being) can have *perceptions.* We assume that each of the experiments ω_k, orderly for $k \in \{1, 2, \dots, n\}$, can be associated with two values: *ability* of the individual $I_k = I(\omega_k)$ to deal with the options of the k-th experiment, and the *constraints* $H_k = H(\omega_k)$ of the environments to disable it. For every k product $L_k = I_k H_k$ we call some *freedom* of an individual in relation to the experiment ω_k, and the sum of freedoms

$$L = L_1 + L_2 + \dots + L_n, \quad (2.11)$$

is the *information of perception* of the given individual.

In the information theory I am talking about, every part of the matter is a type of information. Consistent with that, there is information about each individual freedom as well as abilities or restrictions. Hence the initial names for L libertas, I intelligence, and H hierarchy, with a restriction to living beings, but it is clear from the generality of the theory itself that information of perception also applies to inanimate beings. The basic idea was that greater intelligence (ability) develops in such a way that it seeks more freedom (option, information) and less restrictions ($I = L/H$). Thus we come to the information of perceptions formally as a scalar product of unitary spaces.

For example, when in Heisenberg's relations of uncertainty, we replace the indeterminacy of the momentum Δp_x with freedom, and the indeterminacy of the position Δx with a constraint, both along the abscissa, then their product, $\Delta p \Delta x \ge L_x$, is the freedom of a given particle relative to a given experiment. The smallest value of the mentioned freedom L_x is the quantum of action of a given particle under given conditions.

In general, we get a formalism that equates freedom, action and information. Then we write

$$L = \sum_k \lambda_k \Delta p_k \Delta x_k = \sum_k \ln \exp[\lambda_k \Delta p_k \Delta x_k], \quad (2.12)$$

where the coefficients $\lambda_k \in \mathbb{C}$ are chosen according to the applications. In particular, the wave functions of quantum mechanics ψ_k are types of information perception and, as we know, their linear combinations

$$\psi = \alpha_1\psi_1 + \alpha_2\psi_2 + \dots + \alpha_n\psi_n, \quad (2.13)$$

where the scalar $\alpha_k \in \mathbb{C}$ defines the probability k-th observable, are the vectors representations of quantum states.

[2]A functional is a function that maps vectors to scalars.

Consistently (2.12), we interpret "$\exp[-\lambda_k \Delta p_k \Delta x_k]$" as (some) average probability of information "$-\ln\exp[\lambda_k \Delta p_k \Delta x_k]$", and then "$\exp[\lambda_k \Delta p_k \Delta x_k]$" as the corresponding (mean) number of "equally likely" results of the experiment ω_k.

Information of perception reaches its lowest value when the given individuals are inanimate beings, when their trajectories satisfy the Euler-Lagrange equations and the *principle of least action* of physics. Otherwise, we have physical systems with excess action, that is, information whose movements are not solutions of the mentioned equations.

2.2.2 Schwarz's Inequality

In the algebra of unitary operators a special role is played by the following inequality proved by *Cauchy*[3] 1821 in the case of the space X^n, which in the special case $n = 3$ consequence of one identity proved by *Lagrange*[4] in 1773, and for the space of functions it was proved by *Bunyakovsky*[5] 1859 and *Schwarz*[6] 1885. We use the usual notation for the absolute values of the number, i.e. the norm of the vector $|x| = \sqrt{\langle x|x\rangle}$.

Theorem 2.2.1 (Schwarz's inequality)**.** *For arbitrary vectors x and y of the unitary space X the inequality holds*

$$|\langle x|y\rangle| \le |x||y|, \tag{2.14}$$

where the equality is valid iff[7] *the vectors x and y are linearly dependent.*

Proof. We start from the obvious inequality:

$$0 \le |\langle x|x\rangle y - \langle y|x\rangle x|^2 =$$

$$= |\langle x|x\rangle y|^2 + |\langle y|x\rangle x|^2 - \langle\langle x|x\rangle y|\langle y|x\rangle x\rangle - \langle\langle y|x\rangle x|\langle x|x\rangle y\rangle$$

$$= |x|^4|y|^2 + |\langle y|x\rangle|^2|x|^2 - |x|^2\langle x|y\rangle\langle y|x\rangle - |x|^2\langle y|x\rangle\langle x|y\rangle$$

$$= |x|^2(|x|^2|y|^2 - |\langle x|y\rangle|^2),$$

and hence $|\langle x|y\rangle| \le |x||y|$ за $x \ne 0$. But this relation also holds for $x = 0$. On the other hand, the equals sign gives $\langle x|x\rangle y - \langle y|x\rangle x = 0$, from which follows the dependence of the vectors x and y. This proves the theorem. □

In the case of the probability distribution vector, as in the case of quantum mechanics superposition, it will be $|x| = |y| = 1$, the theorem 2.2.1 says that $|\langle x|y\rangle| \le 1$ and that the equality holds if the vectors are dependent. In case the vectors are independent, their scalar product gives less probability which mean higher total information.

In other words, the dependent vectors x and y of the unitary space X by the scalar product $\langle x|y\rangle$ are so conjugated that they have less information. This is a reinforcement of my recent post which I added to the book "Minimalism of information" (see [2]), otherwise mostly completed in summer 2019 and unpublished until summer 2020. Not only in the case of fermions but also beyond, information of perception $\langle x|y\rangle$ *merged* vectors x and y is smaller than the sum of their individual information.

[3] Augustin-Louis Cauchy 1789-1857, French mathematician.
[4] Joseph-Louis Lagrange 1736-1813, Italian mathematician.
[5] Viktor Bunyakovsky 1804-1889, Russian mathematician.
[6] Hermann Schwarz 1843-1921, German mathematician.
[7] iff – if and only if

The knowledge of association, combined with the *principle of information*, that nature is stingy with information emissions, has various consequences. For example, information tends to come together as if guided by some attractive force. Quantum entanglement as a special type of coupling is also a state of interdependence, and it is already known and recognized in physics.

It is less well known that an electron in an atom can be spoken of as a kind of coupling, so it needs to add energy to get rid of that interconnection. Also, that the player in the *Nash*[8] equilibrium[9] the kind of connection from which only with additional risk and tactics can seek a way out, or the living being adapted to its environment is in association from which it can be extracted only with additional freedoms.

On the other side of inequality (2.14), there is the question of the minimum of the scalar product, i.e. the maximum of the information of the alleged association. We know from algebra that the smallest value of $|\langle x|y\rangle|$ achieves with zero scalar product when the vectors x and y are orthogonal. In the real world, this is not possible, but it is if one of the vectors belongs to the real world and the other to the pseudo-real world. This leads us to *Everett's* "many-worlds" which I also called *pseudo-realities*, or "parallel realities". This last expression comes from the appearance of decimals of the number $\pi = 3,14159265359\dots$ which appear as random numbers, but in mathematics we call them pseudo-random.

In addition to the usual examples of the norm, the intensity of the vector, there is also the potential (forces, energies, actions, information). If the direction of the vector then determines the gradient of the norm, then the spaces of the constant norm are one-dimensional. Without tension there is no field of forces, no exchange of energy, action, emission of information. However, the states of greater and lesser homogeneous tension in information theory cannot be the same, but that is another part of this story.

2.2.3 Normed Space

The norm of the vector seems to "melt" by addition, or let's say the potential of the sum is less than the sum of the potentials. In the second interpretation, each side of the triangle is smaller than the sum of the other two and larger than their difference. This is the content of the following theorem.

Theorem 2.2.2 (Triangle inequality). *За произвољне векторе важи*

$$|x+y| \le |x| + |y|, \quad x, y \in X,$$

where the equality holds iff the vectors are linearly dependent.

Proof. We use Schwarz's inequality:

$$|x+y|^2 = \langle x+y|x+y\rangle = \langle x|x\rangle + \langle y|y\rangle + 2\Re\langle x|y\rangle \le$$

$$\le |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2.$$

Accordingly $|x+y| \le |x| + |y|$. □

[8]John Forbes Nash Jr. 1928-2015, American mathematician.

[9]Nash equilibrium – (in economics and game theory) a stable state of a system involving the interaction of different participants, in which no participant can gain by a unilateral change of strategy if the strategies of others remain unchanged.

Put $z = x + y$ and have $|x| = |z + (-y)| \le |z| + |y|$ и $|y| \le |z| + |x|$, so with the previous inequality we get

$$|x| - |y| \le |x \pm y| \le |x| + |y|, \quad x, y \in X. \tag{2.15}$$

This completes the theorem we call the triangle inequality. From the previous two theorems we conclude that the function $x \to |x|$, which to vector x associates its norm, number $|x|$, has the following properties.

The vectors $x, y \in X$ and the scalar $\lambda \in \Phi$ have the properties *vector norm*:

1. $|x| \ge 0$,
2. $|x| = 0 \iff x = 0$,
3. $|\lambda x| = |\lambda||x|$,
4. $|x + y| \le |x| + |y|$.

A vector space in which each vector x is associated with the number $|x|$ with all four specified properties is called a normalized vector space.

The normalized space is a special case of unitary ones, where the norm of the vector is expressed by the scalar product $|x| = \langle x|x\rangle^{1/2}$. Hence:

$$|x + y|^2 = \langle x + y|x + y\rangle = \langle x|x\rangle + \langle x|y\rangle + \langle y|x\rangle + \langle y|y\rangle$$

$$|x - y|^2 = \langle x - y|x - y\rangle = \langle x|x\rangle - \langle x|y\rangle - \langle y|x\rangle + \langle y|y\rangle,$$

so by addition we find

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2. \tag{2.16}$$

This is the well-known *parallelogram relation* spanned by the vectors x and y, with diagonals of their sum and difference.

We say that the vector e is *normalized* or the unit vector, if $|e| = 1$. We say that the vectors x and y are (mutually) *orthogonal* or perpendicular when $\langle x|y\rangle = 0$. A set of vectors is said to be orthogonal if any two vectors of that set are perpendicular to each other. A set of vectors is *orthonormal* if it is orthogonal and if each of its vectors is normalized.

2.2.4 Orthonormalization of vectors

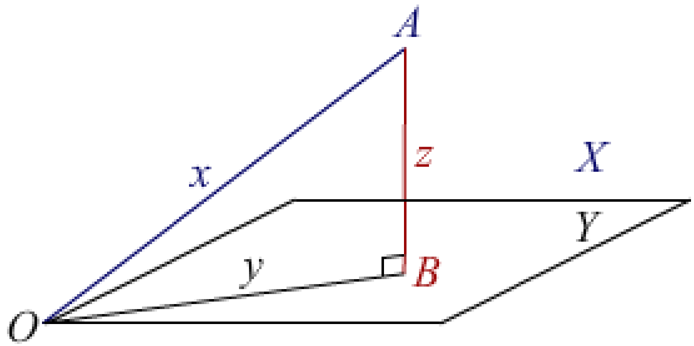
In Euclidean geometry, we consider the notion of the orthogonality of a vector quite clear, and then we imitate it even where it is no longer so obvious. Thus, we consider the vectors x and y of the unitary space X to be orthogonal (mutually perpendicular) if their scalar product is zero, i.e.

$$x \perp y \iff \langle x|y\rangle = 0, \quad x, y \in X, \tag{2.17}$$

where the scalar product in algebra is usually written $\langle x|y\rangle = x \cdot y$. Therefore, the zero vector ($x = 0$) is perpendicular to each vector. In the figure 2.1 the vector $z = \overrightarrow{BA}$ is perpendicular to the plane Y, which we can write $z \perp Y$, so then x is perpendicular to each vector equal to Y.

From the same image we see that $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$, or shorter $x = y + z$. We say that the vector $y \in Y$ is the projection of the vector $x \in X$ on the subspace $Y \subseteq X$, where $|y| = |x| \cos\angle(x, y)$. Let us denote the corresponding unit vectors by the lower indices, say:

$$e_x = \frac{x}{|x|}, \quad e_y = \frac{y}{|y|}. \tag{2.18}$$



Slika 2.1: Projection of the vector $x \in X$ onto the subspace Y.

These are unit vectors of direction x and y, so:

$$z = x - \langle x|e_y\rangle e_y = x - \frac{\langle x|y\rangle y}{|y|^2} = \frac{\langle y|y\rangle x - \langle x|y\rangle y}{|y|^2},$$

and hence the record

$$z = \frac{1}{\Gamma(y)}\begin{vmatrix} \langle y|y\rangle & y \\ \langle x|y\rangle & x \end{vmatrix}, \quad |z|^2 = \langle z|z\rangle = \frac{\Gamma(y,x)}{\Gamma(y)}. \tag{2.19}$$

Here we use the usual notation for the *Gram determinant* of vector:

$$\Gamma(x_1) = |\langle x_1|x_1\rangle|, \dots, \Gamma(x_1, x_2, \dots, x_n) = \begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle & \dots & \langle x_1|x_n\rangle \\ \langle x_2|x_1\rangle & \langle x_2|x_2\rangle & \dots & \langle x_2|x_n\rangle \\ \dots \\ \langle x_n|x_1\rangle & \langle x_n|x_2\rangle & \dots & \langle x_n|x_n\rangle \end{vmatrix}. \tag{2.20}$$

As is known, with the help of similar determinants it is possible to orthogonalize an arbitrary set of vectors.

Let x_1, x_2, x_3 be three linearly independent vectors in space. Based on (2.19) we find the first two of the vectors, and then we calculate the third:

$$e_1 = \frac{x_1}{\sqrt{\Gamma(x_1)}}, \quad e_2 = \frac{\begin{vmatrix} \langle x_1|x_1\rangle & x_1 \\ \langle x_2|x_1\rangle & x_2 \end{vmatrix}}{\sqrt{\Gamma(x_1)\Gamma(x_1,x_2)}}, \quad e_3 = \frac{\begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle & x_1 \\ \langle x_2|x_1\rangle & \langle x_2|x_2\rangle & x_2 \\ \langle x_3|x_1\rangle & \langle x_3|x_2\rangle & x_3 \end{vmatrix}}{\sqrt{\Gamma(x_1,x_2)\Gamma(x_1,x_2,x_3)}}. \tag{2.21}$$

These are orthonormal vectors of the same space.

I assume that we are familiar with the primal views on the basis of vector space, then Gram-Schmidt's procedure of orthogonalization, and the following position on orthogonalization, which I state without proof.

Proposition 2.2.3 (Orthogonalization)**.** *Each $x_1, x_2, \dots, x_n, \dots \in X$ finite and countable (if exists) series of linearly independent vectors of unitary space X can be replaced (orthonormalized) by a series of orthonormal vectors $e_1, e_2, \dots, e_n, \dots$ which spans the same space as the starting series.*

2.2.5 Vector projection

The figure 2.1 is an example of an Euclidean space X with vector $x = \overrightarrow{OA}$ and subspace $Y \subset X$ with vector $y = \overrightarrow{OB}$ such that the vector $z = x - y = \overrightarrow{BA}$ is perpendicular to $y \in Y$. We have an analogous situation in unitary spaces, which is discussed in the following theorem.

Theorem 2.2.4 (Projection)**.** *If Y is a finally dimensional subspace of the unitary space X then each vector $x \in X$ can be uniquely written in the form*

$$x = y + z$$

where $y \in Y$ and $z \perp Y$.

Proof. Existence. Since Y is finally dimensional, there exists a base $y_1, y_2, \dots, y_n$ of the subspace Y, so there is also an orthonormal base $e_1, e_2, \dots, e_n$. For a given vector $x \in X$ the vectors are completely specified:

$$y = \sum_{k=1}^{n} \langle x|e_k\rangle e_k, \quad z = x - y.$$

It is obvious $y \in Y$, and from

$$\langle z|e_j\rangle = \langle x - y|e_j\rangle = \langle x|e_j\rangle - \sum_{k=1}^{n} \langle x|e_k\rangle\langle e_k|e_j\rangle = 0$$

follows $z \perp e_1, e_2, \dots, e_n$. Hence, $z \perp Y$.

Unambiguity. Let it be

$$x = y + z = y' + z' \quad (y, y' \in Y;\ z, z' \perp Y).$$

Then:

$$y - y' = z' - z,$$
$$\langle y - y'|y - y'\rangle = \langle y - y'|z' - z\rangle = 0,$$

so $y - y' = 0$, i.e. $y = y'$, and then also $z' = z$. □

We know that space-time is at least 6-dimensional[10] and that it consists of three dimensions of space and three dimensions of time, of which we can observe only four, three spatial and one temporal. The rest of the 4-D real space-time from the 6-D space-time in the texts I wrote in previous years I usually called *parallel reality*, but you can also call them *many worlds* after Everett, or *pseudo-reality*, which would be more principled, say, than *multiverse*, which is an idea that in one form or another dates back to the time of ancient Greek mythology. Unlike other "multiverses", normal physical communication with "parallel reality" is not possible without violating the law of conservation.

We are not dealing here with the proofs of "many worlds", but we are only assuming them in the form that they contain invariant laws of physics, i.e. quantum mechanics, which means that we can consider them to be one Hilbert 6-D vector space X within which Y is our 4-D space-time.

The position on orthogonalization then says that "many worlds" can be spaned into orthogonal observable "realities and parallel realities", and the projection theorem further

[10]Space-time: `https://www.academia.edu/40603049/Space_and_time`

speaks of the possibilities of orthogonal projection of any vector (quantum state) of that space onto reality, moreover, about its uniqueness.

On the other hand, from the point of view of quantum mechanics, the orthogonal projection of the vector x on y determines the probability that the quantum state x exhibits the property y. The scalar product of two vectors is a measure of their conjugation, or adaptation (fidelity), which is also expressed by probability. Therefore, the orthogonal vector (z) to the real vector ($y \in Y$) is always at least pseudo-real (Figure 2.1).

2.3 Metric space

Metric space is a set M on which the distance is defined, the metric $d(a,b)$ between arbitrary elements $a, b \in M$, i.e. the function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x_1, x_2) \ge 0$ – non-negativity,
2. $d(x_1, x_2) = 0 \iff x_1 = x_2$,
3. $d(x_1, x_2) = d(x_2, x_1)$ – symmetry,
4. $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$ – triangle inequality,

for all $x_1, x_2, x_3 \in X$. We call such a function the *metric* of the given space.

It is easy to check that the function $d(x,y) = |x - y|$ for x and y from the normed (normalized) space has metric properties. As metric space does not have to be even linear, it is a broader concept than normed space. We call its elements points.

When two metric spaces X and Y are supplied with metrics d_X and d_Y so that $Y \subset X$ and $d_X = d_Y$ for each pair of points from Y, then for Y we say that is the metric *subspace* from X. Each part of the metric space X can be understood as its metric subspace in which the metric is called *induced* by the metric X.

Two metric spaces X and Y are *isometric* if there is a mutually unique (bijective) mapping f between them such that for each pair of points $x_1, x_2 \in X$ holds

$$d_Y[f(x_1), f(x_2)] = d_X(x_1, x_2). \tag{2.22}$$

Such a mapping f is an isometry between the spaces X and Y.

The *distance* between the sets $A, B \subset X$ of metric space is defined by *infimum*, the so-called maximum lower limit

$$d(A,B) = \inf_{x \in A, y \in B} d(x,y), \tag{2.23}$$

and if the set B consists of only one point b, we are talking about the distance of the point to the set

$$d(b,A) = d(\{b\}, A) = \inf_{x \in A} d(b,A). \tag{2.24}$$

Diameter of the set $A \subset X$ is defined by *supremum*, the so-called minimum upper limit

$$d(A) = \sup_{x,y \in A} d(x,y). \tag{2.25}$$

We say that the set A is *limited* if it has a finite diameter.

2.3.1 Closed and open sets

Sphere with center a of radius ρ form those points $x \in X$ that satisfy the condition $d(a,x) = \rho$. A set of points $x \in X$ for which is orderly:

$$d(x,a) < \rho, \quad d(x,a) \le \rho, \tag{2.26}$$

it is called an open or closed *ball* with center at point a and radius ρ.

In the textbook of functional analysis [15], an open ball is denoted by $K]a,\rho[$, closed by $K[a,\rho]$, and when it does not matter whether it is open or closed by $K(a,\rho)$. The idea is generalized to the space $\mathbb{R}^n$ with arbitrary $n \in \mathbb{N}$, so the set $A \subset X$ is said to be open if for every $x \in A$ there is some $\rho > 0$ so $K]x,\rho[\subset A$. Thus, "balls" are arbitrary in dimensions n, and are even intervals, for $n = 1$.

An open ball is the simplest open set, because for $x \in K]a,\rho[$ is $d(x,a) < \rho$, so $K]x,\rho - d(x,a)[\subset K]a,\rho[$. An empty set is open, because it does not contain any point to which a certain condition would be imposed. It is also clear that the whole space is an open set. In this text, we will denote open sets with the letter G and indexes.

The union of any collection of open sets $\{G_i\}_{i\in I}$ is *open set*. Namely, when $x \in \bigcup_{i\in I} G_i$, then for some $i_0 \in I$, $x \in G_{i_0}$. But since, assuming G_{i_0} is an open set, there exists a ball $K]x,\rho[\subset G_{i_0}$; the more so $K]x,\rho[\bigcup_{i\in I} G_i$, which should have been shown.

The intersection of finally many open sets is an open set. Namely, from $x \in \bigcap_{i=1}^n G_i$ follows $x \in G_i$ for every $i = 1,2,\dots,n$. Since all these sets are open, there are positive numbers ρ_i, for all indices $i = 1,2,\dots,n$, so that $K]x,\rho_i[\subset G_i$. If we put $\rho = \min\{\rho_1,\rho_2,\dots,\rho_n\}$, the ball $K]x,\rho[$ will be in each of the sets G_i, and therefore in their intersection. This, then, is an open set.

The set $A \subset X$ is *closed* if its complement in relation to the whole space is an open set. For example, a closed ball is a closed set, each finite set is closed, each sphere is a closed set. We will denote a closed set by the letter F.

Due to the duality of the notions of open and closed sets, the following paragraphs immediately follow from the previous ones. The whole space and the empty set are closed sets. The intersection of any collection of closed sets is a closed set. The union finally many closed sets is a closed set.

Like the spacing $]a,b] \subset \mathbb{R}$ most sets are neither open nor closed. The properties of only closed sets belong to the physical substance. Their intersections and unions are final sets that are closed to the above, and especially an empty set and the whole space are both the physical (concrete) and non-physical (abstract) things.

2.3.2 Neighborhood set

Neighborhood of the set $A \subset X$ is any set V that contains one open set in which the set A lies. Therefore, the set A is open if and only if it is the neighborhood of each of its points. Namely, if A is an open set, it is by definition the neighborhood of each of its points. Conversely, if A is the neighborhood of its arbitrary point x, then each of these points corresponds to an open set such that $x \in G_x \subset A$. But from

$$A = \bigcup_{x\in A}\{x\} \subset \bigcup_{x\in A} G_x \subset A,$$

follows

$$A = \bigcup_{x\in A} G_x,$$

so since the union of open sets is an open set, it is a A open set.

The point x is the *interior point* of the set A if A is its neighborhood. The collection of all interior points of the set A forms a *interior* of A which we denote by $\mathring{A}$, a circle above the mark of the set. Obviously, it is $\mathring{A} \subset A$. The interior of the interval (a,b) is the open interval $]a,b[$. The interior of a set of rational numbers is an empty set.

It is true that $\mathring{A}$ is the largest open set contained in A, so the open sets are characterized by $\mathring{A} = A$. Also, the interior of the interior is equal to the interior of a given set. In the case of two sets, if the first is a subset of the second, then the interior of the first is a subset of the interior of the second. The interior of the intersection of sets is equal to the intersection of their interiors. See the proves of these statement in [15].

The point x is the *adherent point* of the set A if at least one point from A lies in each of its neighborhoods. The collection of adherent points of the set A forms a *adherence* of A which denote by $\bar{A}$. Obviously $A \subset \bar{A}$, the adherence of the interval $(a,b) \subset \mathbb{R}$ is the closed interval $[a,b]$, the adherence of the set of rational numbers is the set of all real numbers.

Note that if x is not an internal (adherent) point of a set then it is an adherent (internal) point of a complementary set. Due to this duality and previous attitudes, the following is also valid. The set $\bar{A}$ is the smallest closed set containing A, for closed sets is $\bar{A} = A$, $\overline{(\bar{A})} = \bar{A}$, from $a \subset B$ follows $\bar{A} \subset \bar{B}$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

The *boundary* point of the set is the one that is at the same time an adherent point of both the set and its complement. The collection of all boundary points of the set A forms boundary (edge) of A. Thus, boundary of the interval $(a,b) \subset \mathbb{R}$ made the points a and b. The boundary of a set of rational numbers is a set of real numbers. The boundary of each set is a closed set, because it is by definition the intersection of the adherent (adhesion) of the set and the complement of the set, and they are closed, so their intersection is closed.

2.3.3 Accumulation point

The point x is the *isolated point* of the set A if there is an neighborhood of the point x in which, except x, there are no other points from A. The point x is the cluster or *accumulation point* of the set A if at least one point from A different from x lies in each of its neighborhoods.

The collection of accumulation points of the set A forms a *derived set* of A which we denote by A'. The accumulation point may or may not belong to the set. For example, the set $\{1/n\}$, for $n = 1,2,3,\dots$, does not have zero, its only point of accumulation. A derived set of rational numbers is a set of real numbers.

Note that in the general case $(A')'$ is not equal to A'. For example, the mentioned set $\{1/n\}$ is derived into $\{0\}$ and derived to this is an empty set.

However, for an arbitrary set A, $\bar{A} = A \cup A'$ holds. Namely, if $x \notin \bar{A}$ then there is a neighborhood of the point x in which there are no points from A. Then x does not belong to either A or A', nor to their union. Conversely, if $x \notin A \cup A'$, there is a neighborhood in which there are no points from A, and it cannot lie in $\bar{A}$.

That the set A is closed if and only if all the points of accumulation belong to it, i.e. if $A' \subset A$, it is proved like this. If A is a closed set, i.e. $\bar{A} = A$, will be based on the previous $A = A' \cup A$, hence $A' \subset A$. Conversely, from $A' \subset A$ follows $A' \cup A = A$, then $\bar{A} = A$, so A is closed.

In the functional analysis and topology for the set A we say that is *perfect*, if all the accumulation points belong to it and if each of its points is an accumulation point, i.e. if $A' = A$. The closed space in $\mathbb{R}$ is a perfect set, and the set of all real numbers is perfect.

In short, every closed set is a disjoint union of its isolated points and its accumulation points, and that is exactly what we have with physical properties. I have previously argued that the property to which the law of conservation (information, energy) applies cannot be infinitely divisible, it is discrete, and so this knowledge from *functional analysis* about environments can now be viewed as a further discussion of the connections between the physical concrete and abstract mathematical.

2.3.4 Everywhere dense

The set A is *everywhere dense* in the set B, if any point from B is an adherent point of A, i.e. if $B \subset \bar{A}$. In other words, the set A is everywhere dense in B if there is at least one point in A in each neighborhood of any point in B. For example, the set of rational numbers $\mathbb{Q}$ is everywhere dense in the set of real numbers $\mathbb{R}$.

The set $A \subset X$ is *nowhere dense* in X, if its adherence $\bar{A}$ does not contain any balls or, what is the same, if $\bar{A}$ has no internal points. For example, the set of integers $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$. The final union of nowhere dense sets is nowhere dense set, but that does not have to be true for a countable union. For example, a set of rational numbers is a countable union of nowhere dense sets consisting of one point each, but itself is nowhere dense.

The set $A \subset X$ is *first category* in X, if it is the union of the most countable[11] nowhere dense sets from X. Any set that is not the first is by definition *second category* in X.

The space X is of the second category in itself if and only if, no matter how it is presented as a union of the most countable closed sets, $X = \bigcup_{i=1}^{\infty} F_i$, in at least one of (closed) sets F_i lies a ball.

Namely, if there were no ball in any of the sets F_i, all sets F_i would be nowhere dense in X, so X, as the most countable union of nowhere dense sets, would be a set of the first category.

Conversely, if X is not second category in itself, it can be represented as the most countable union of nowhere dense sets, $X = \bigcup_{i=1}^{\infty} A_i$, but then is also $X = \bigcup_{i=1}^{\infty} \bar{A}_i$, i.e. X can be represented as the most countable union of closed sets, none of which contains a ball. This proves the other half of the claim.

Applied to physics, the sequence of events realized in time is a discrete set; the set of all their possibilities is a continuum. Unrealized possibilities belong to the multiverse of *parallel realities*[12], a dense set everywhere without direct physical communication with reality.

2.3.5 Connected space

The space X is *connected* if it cannot be represented as a union of two nonempty disjoint open sets from X. Conversely, the space X is not connected, i.e. *disconnected*, if there are open sets $G_1 \neq 0$, $G_2 \neq 0$, $G_1 \cap G_2 = \varnothing$, such that $X = G_1 \cup G_2$. Since G_2 is a complement of G_1 and G_1 is a complement of G_2, G_1 and G_2 are both closed sets, so in the definition of connectivity it is possible to replace the word "open" with, "closed" and get the following two definitions.

A space X is connected if in it, apart from an empty set and the whole space, there is no other set that would be both open and closed at the same time. Namely, if A were such a set, such would be the complement of that set, $C(A)$, so $A \cup C(A) = X$ would not be a connected space.

[11] not more than countably infinite

[12] I do not limit the "multiverse" to only this type.

The set $A \subset X$ is connected if such is observed for itself as a metric space. For example, any connected set on a real line is reduced to an interval of arbitrary type, finite or infinite.

Connection sets are interesting to us because physical space-time is like that. It cannot be presented as a union of its two non-empty disjunctive sets, because that would be a direct mixing of the physical and the abstract. That is why a vacuum is conditionally called a physically "infinite" set, and a universe "finite".

2.3.6 Cantor set

Any non-empty open set in $\mathbb{R}$ is a union of at most countable many disjoint open intervals.

Namely, let the given point $x \in G \subset X$ be in an open set. Then there is an open interval that is, $x \in I \subset G$, which we can always choose so that its ends are rational numbers. If we to each point assign one such interval, we will have a countable collection of intervals. In the collection of these intervals we introduce the equivalence relation, $I' \sim I''$, if there is a disjoint finite series of intervals $I_1 = I', I_2, \dots, I_n = I''$ from that collection. The corresponding set of equivalence classes is obviously countable.

It immediately follows from the previous paragraph: Every non-empty closed set in $\mathbb{R}$ is obtained when the most countable open intervals are removed from the real line.

These two attitudes tell us both about open and closed sets and about the nature of the infinite (abstract) and finite (physical), because every finite set is open, and every property of physical information is finally divisible. If we further abstracted and shared those smallest pieces of physical information, we would get the most countable many open sets.

The simplest closed sets in $\mathbb{R}$ are closed intervals, isolated points, and unions of finally many that. They are complements of open sets which are, therefore, a union of finally many disjoint open intervals. An interesting example of such a union is the so-called *Cantor's set* which I will now describe.

Let F_0 be the closed interval $[0,1]$. We remove from F_1 the open interval $]1/3, 2/3[$ and mark what remains with $F_1 = [0, 1/3] \cup [2/3, 1]$. Remove the middle thirds in these two intervals and mark what remains with $F_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$. We continue this procedure without end; we get a series of closed sets $F_n \supset F_{n+1}$, where F_n is a union of 2^n closed intervals, each of which is 3^{-n} of length.

Cantor set F is defined by

$$F = \bigcap_{n=1}^{\infty} F_n. \tag{2.27}$$

Based on the previous views, we see that it is closed, and the points

$$0,\ 1,\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{9},\ \frac{2}{9},\ \frac{7}{9},\ \frac{8}{9},\ \dots, \tag{2.28}$$

as the ends of the removed open intervals, they obviously belong to the set F, but they are not the only points of that set.

It is shown (see [15]) that Cantor's set F is formed by those and only those points of the interval $[0,1]$ which written *triad*, in the base of three digits 0, 1 and 2, do not have unit. It has a cardinal continuum number. The points (2.28) of the Cantor set are called *points of the first kind* and there are countable of them, and all other points of this set are called *points of the second kind*. Cantor's set F is a perfect set, and its points of the first kind are the points of accumulation of points of the first kind. Cantor set is nowhere dense.

For example, if all quantum states consisted of *qutrit*, a superposition of three outcomes (three mutually orthogonal quantum states), we could form their Cantor set and the corresponding triad records. After the realization of the outcome, we join the number 1. All these numbers that contain only units make a countable set of real outcomes, and those with the other two digits, 0 or 2, a not-countable set of parallel realities. We get the same with more complex superposition and corresponding interpretations of Cantor's set, more complex interval divisions and larger *number bases*.

Cantor set is an interesting example for explaining the coverage of an open set with countably closed cubes. Namely, by *closed cube* in $\mathbb{R}^n$ whose sides are parallel to the coordinate planes we mean the set

$$\{x = (\xi_\nu) : \ \alpha_\nu \le \xi_\nu \le \alpha_\nu + \alpha, \ \nu = 1, 2, \dots, k\}. \tag{2.29}$$

Each nonempty open set in $\mathbb{R}^n$ ($k \ge 2$) can be represented as a countable union of closed cubes that two and two have no common interior points and whose sides are parallel to the coordinate planes. This attitude[13] indicates the possibility of forming the abstract using the physical.

2.3.7 Real axis

The following two statements, very important for functional analysis, are valid on the set of real numbers, which can be transferred to general metric spaces spanned (built) by such, continuous coordinate axes. The first theorem is *Bolzano*[14] and *Weierstrass*[15].

Theorem 2.3.1 (Bolzano-Weierstrass)**.** *Each bounded infinite set in* $\mathbb{R}$ *has at least one accumulation point.*

Proof. Let the points of the set lie in the interval $[a, b]$, $b - a < +\infty$. Let us divide this into two parts and let $[a_1, c_1]$ be one of those parts that has infinitely given points. Let's divide the chosen ones again and choose the part with infinitely many points. Continuing this procedure, we get through the inserted intervals $[a_n, c_n]$ ($n = 1, 2, 3, \dots$), each of which contains infinitely many points. A series of intervals tends to one point which is the point of accumulation of a given series of points. □

For example, the consequence of this theorem is the claim that every continuous function on a closed bounded interval is bounded. Another example is Weierstrass's view that every continuous function can be uniformly approximated by polynomials. This means that the set of polynomials is everywhere dense in the set $C[a, b]$ of continuous functions on the given interval with the metric

$$d(x, y) = \max_{a \le t \le b} |x(t) - y(t)|. \tag{2.30}$$

A step away from there is known theorems on the development of periodic functions into trigonometric and other series. They make it possible to represent the uncertainty of quantum particles.

Theorem 2.3.2. *Let* F *be a closed set in* $\mathbb{R}$*, bounded on the right (left) side. Then* $\sup F$ *(*$\inf F$*) belongs to the set.*

[13]Proof is in [15].
[14]Bernard Bolzano (1781-1848), a Czech mathematician of Italian descent.
[15]Karl Weierstrass (1815-1897), a German mathematician, often referred to as the "father of modern analysis".

Proof. Let $b = \sup F$ and assume that $b \notin F$. Based on the definition of supremum, for every $\epsilon > 0$ there is a point $x \in F$ such that $b - \epsilon < x < b$, i.e. in each neighborhood of the point b lies at least one point x from F and $x \neq b$, because b does not belong to F. So b is the accumulation point of F and does not belong to F, and that is a contradiction, because F is a closed set. □

Two real lines span (crucify) a plane of complex numbers which are scalars of Hilbert vector space, and whose representation is quantum mechanics. That is one of the reasons why we need these famous attitudes. We get the second when we project the space-time of the universe onto its space. The evolution of a material event becomes infinite through the points of space whose point of accumulation we interpret as the point of accumulation of the past. This interpretation will gain in importance with the eventual acceptance of the idea of a space (universe) that remembers its history.

2.3.8 Separable space

The metric space X is *separable* if it contains the most countable[16] everywhere dense set of points.

For example, the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) is separable, whose elements are ordered strings of n real (or complex) numbers. Thus the points $x = (\xi_1, \xi_2, \dots, \xi_n)$, $y = (\eta_1, \eta_2, \dots, \eta_n)$ and $z = (\zeta_1, \zeta_2, \dots, \zeta_n)$ are elements of the space $\mathbb{R}^n$. Metrics in this space can be introduced in several ways:

$$d_p(x,y) = \left(\sum_{k=1}^{n} |\xi_k - \eta_k|^p \right)^{1/p}, \tag{2.31}$$

where for $p = 2$ we get the classical Euclidean distance, and in the limit case $p \to \infty$ we have $d_\infty(x,y) = \max_{1 \le k \le n} |\xi_k - \eta_k|$. When we move on to the boundary cases $n \to \infty$ in these examples, we also get the separable space ℓ_p, for $1 \le p < \infty$, where $\ell_1 = \ell$. The space of continuous functions of the $C[a,b]$ metric (2.30) is also separable.

A collection of open sets in a metric space forms a *base* of space if each nonempty open set of spaces can be represented as a union of sets from the base. A space is said to be the most countable base if there is a base in it with the most countable many elements. It is easily proved[17] that space of the most countable base is if and only if it is separable.

A collection of sets is one *covering* of a set if each point of a given set lies in at least one set of the collection. When it comes to the collection of open sets, then we are talking about open coverage. When we have one open coverage of a separable space, then one of the most countable covers of a given space can be extracted from it [18].

2.3.9 Series of points

Let X be a metric space and $x_n \in X$ $(n = 1, 2, 3, \dots)$ be a series of points, labeled (x_n). We call the point $x \in X$ the *limit value* of the sequence (x_n), and for the sequence we say that *converges* to that point if $d(x_n, x) \to 0$ when $n \to \infty$. Then we write $x_n \to x$, or $\lim_{n\to\infty} x_n = x$. In other words, if each neighborhood $V_x \subset X$ of the point x corresponds to some natural number n_0 that

$$n \ge n_0 \implies x_n \in V_x, \tag{2.32}$$

[16] not greater than countably infinite
[17] See in any better textbook of functional analysis.
[18] LCT: `http://mathonline.wikidot.com/the-lindeloef-covering-theorem-in-euclidean-space`

then $x_n \to x$ when $n \to \infty$.

Each convergent sequence is *limited.* Namely, due to $d(x,x_n) < \varepsilon$ for $n \geq n_0$ almost all[19] Points of the set $\{x_n\}$ are in the ball $K]x,\varepsilon[$. For a sufficiently large ρ, all the points of that set are in the ball $K]x,\rho[$.

For every strictly ascending sequence of natural numbers $n_1 < n_2 < n_3 < \dots$ the sequence (x_{n_k}) is *partial sequence* of the sequence (x_n). The point x is the *adherent value* of the sequence (x_n) if there is a partial string (x_{n_k}) that converges to x.

A necessary and sufficient condition that x is an adherent value of the sequence (x_n) is that every neighborhood V_x of the point x and every natural number m corresponds to a natural number $p > m$ so that $x_p \in V_x$.

Namely, the condition is necessary because for the adherent value of a given sequence there is a partial sequence, subsequence (x_{n_k}) which converges to x. In every neighborhood V_x lie almost all members of a subsequence, so no matter how large the natural number m is, there will be a large enough n_k so that $n_k > m$ and $x_{n_k} \in V_x$. The condition is sufficient, because if it is fulfilled from the given sequence, we will extract the subsequence by total induction: $n_1 = 1$, and when we reach n_{k-1} we will choose n_k as the first natural number greater than n_{k-1} for which $d(x,x_{n_k}) < 1/k$. So $\lim_{k\to\infty} d(x,x_{n_k}) = 0$, i.e. x is the adherent value of the given sequence.

Each adherent value of the sequence, string (x_n) is one adherent point of the set $\{x_n\}$, but not the other way around. For example, the sequence $1/n \in \mathbb{R}$ has zero as a single adherent value, but all elements of that set together with zero are adherent points. The difference between the adherent value of the sequence (x_n) and the accumulation points of the set $\{x_n\}$ is similar. Each accumulation point of that set is an adherent value of the corresponding sequence, but not the other way around, as we see in the example of the sequence $1,1,1,\dots$ from $\mathbb{R}$ which has a unit as an adherent value, but the corresponding set has only one element and has no accumulation point.

In short, if we denote by $\mathcal{U}(x_n)$ the collection of adherent values of the sequence (x_n), then the implications apply

$$\{x_n\}' \subseteq \mathcal{U}(x_n) \subseteq \overline{\{x_n\}}, \tag{2.33}$$

where prim means *derived set*, a collection of accumulation points of a given set, and top line *adherence*, a collection of adherent set points. Both of these inclusions can be strict.

The set of adherent values of an array, sequence is a closed set[20]. This means that the collection of $\mathcal{U}(x_n)$ adherent values of the sequence (x_n) includes all its accumulation points; if z is the accumulation point of that collection, it is also the accumulation point of the set $\{x_n\}$. From the first inclusion (2.33) follows the closeness of the set $\mathcal{U}(x_n)$. Namely, if z is the accumulation point $\mathcal{U}$, for every $\varepsilon > 0$ there will be at least one of the z different point x in $\mathcal{U}$, so that $d(z,x) < \varepsilon$. Since x is an adherent value of the sequence, string (x_n), there is a partial string, subsequence $x_{n_k} \to x$. For a sufficiently large k, $d(x_{n_k},x) < \varepsilon$, which together with $d(z,x) < \varepsilon$ gives

$$d(z,x_{n_k}) \leq d(z,x) + d(x,x_{n_k}) < 2\varepsilon,$$

i.e. z is the accumulation point of the set $\{x_n\}$.

A necessary and sufficient condition that x is an adherent point of the set A is that there exists a series of points, sequence (x_n) from A that converges to x.

[19] Almost all - except them finally many.

[20] These views are well known in functional analysis, so I do not single them out as special theorems.

A necessary and sufficient condition that x is the accumulation point of the set A is that there exists a series of mutually different points in the sequence (x_n) from A that converges to x.

Namely, if x is an adherent point of the set A, then in each ball $K]x, 1/n[$ in order for $n = 1, 2, \dots$ lies at least one point x_n from A. These points x_n, which can all coincide with x, form a string, sequence that converges to x. If x is the accumulation point of the set A, then in each ball $K]x, 1/n[$ a point x_n different from all previously selected points can be chosen. It is clear that the stated conditions are sufficient in both cases.

We will deal more with these notions of functional analysis (and topology) when we need knowledge of the implications and other connections between the concrete and the abstract, reality and pseudo-reality.

2.3.10 Complete space

The string, sequence (x_n) is called *Cauchy* if to any $\varepsilon > 0$ corresponds a natural number n_0 such that $m > n \ge n_0$ implies $d(x_m, x_n) < \varepsilon$.

That each Cauchy sequence is bounded is proved in the following way. If (x_n) is a Cauchy sequence, then for $n = n_0$ $m > m_0 \implies d(x_m, x_n) < \varepsilon$ holds, so almost all members of the sequence (except for finally many of them) are in the ball $K]x_{n_0}, \rho[$.

We prove that every convergent series (sequence) is Cauchy's in the following way. If $x_n \to x$, then for every $\varepsilon > 0$ there is n_0 so that $n \ge n_0$ pulls $d(x, x_n) < \varepsilon$, then for $m > n \ge n_0$ is valid $d(x_m, x_n) \le d(x_m, x) + d(x_m, x) + d(x, x_n) < 2\varepsilon$, which means that (x_n) is a Cauchy series.

The reverse statement of the latter would not be correct. This demonstrates an example of a set of rational numbers with a normally defined interval and series of finite decimal fractions $x_n \to x$ that we get when we write a given irrational number x with approximately the first $n = 1, 2, 3, \dots$ decimals. Such a sequence, of an increasingly accurate irrational number x, Cauchy's is in the set of rational numbers, but does not converge to a rational number.

A metric space is *complete* if each Cauchy series converges in it. The axis of real numbers is one complete space; the axis of rational numbers is not. The metric space X is complete if and only if the intersection of each monotonically decreasing series of closed balls K_n in X, whose radius converge to zero when $n \to \infty$, contains a single point in space. For an explanation and proof of this and the following two paragraphs, see a textbook on metric space (see [17]).

Let X be a complete metric space and let S be some subspace of it. If S is a closed set in X, then S, viewed for itself as a metric space, is also a complete space. Each complete metric space is a *second category* in itself (Baire Category Theorem).

The following paragraph speaks about the completion of the space. Let X be an incomplete metric space. Then there is a complete metric space S, so that one dense subspace S^* is isometric with X. We say that the space S was created *by completing* the space X.

2.3.11 Limit of a sequence

For a series of (x_n) points in a set of real numbers, $\mathbb{R}$, we say that (definitely) *diverges* to $+\infty$ and we write $x_n \to +\infty$ if to each (no matter how large) the real number M corresponds to the natural number $n_0 \in \mathbb{N}$ such that $n \ge n_0$ draws $x_n > M$. Symmetrically means $x_n \to -\infty$.

The sequence of points (x_n) in $\mathbb{R}$ *monotonically* grows if $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$. If instead of "less or equal" there is a relation "less", then it is really or strictly increasing. Symmetrically defined arrays are monotonically and actually, strictly decreasing.

Let S be a non-empty set in $\mathbb{R}$. For a real number m we say that *minor* is set S when $m \le x$ for every $x \in S$. For the number m we say that *infimum* of the set S if m is a minor of the set S and for every $\varepsilon > 0$ there is $x \in S$ such that $x < m + \varepsilon$. In that case we write $m = \inf S$.

Let S be a non-empty set in $\mathbb{R}$. For a real number M we say that *major* is a set S when $M \ge x$ for every $x \in S$. For the number M we say that is *supremum* of the set S if M is a major of the set S and for every $\varepsilon > 0$ there is $x \in S$ such that $x > M - \varepsilon$. Then we write $m = \sup S$.

In other words, the infimum is the largest minor, and the supremum is the smallest major. Every non-empty top-bounded set of real numbers ($\mathbb{R}$) has a supremum, and every bottom-bounded has an infimum.

A monotonically increasing sequence of (x_n), implies real numbers, or converges to $\sup\{x_n\}$ or diverges to $+\infty$ depending on whether it is bounded on the right or not. A symmetrical position is considered to be a monotonically descending sequence. These statements are easily proved as well as the following, the *Weierstrass*[21] statement for real sequences: every bounded sequence of real numbers has at least one *adherent value*.

Let (x_n) be a series of real numbers and let $\mathcal{U}(x_n)$ be a set of its adherent values. *Limes inferior* and *limes superior* of the series (x_n) are defined by:

$$\lim_{n\to\infty} \inf x_n = \inf \mathcal{U}(x_n), \quad \lim_{n\to\infty} \sup x_n = \sup \mathcal{U}(x_n), \tag{2.34}$$

where instead of $\inf\{x_n\}$ we write shorter $\inf x_n$ and similar for superior. Thus $\limsup x_n = +\infty$ if a given series is not bounded on the right, or $\liminf x_n = -\infty$ if a given series is not bounded on the left.

The following claims are also well known, so I only state them without proof. For a real series, string, $-\infty < \limsup x_n < +\infty$, for every $\varepsilon > 0$ there are infinitely many natural numbers, index n_k of a given string, such that $x_{n_k} > \limsup x_n - \varepsilon$, or there exists a natural number n_0 such that $x_n < \limsup x_n + \varepsilon$. The only number that has both of these properties is exactly $\limsup x_n$. Symmetrical statements apply to limes inferior.

Let (x_n) be a series of real numbers. Then:

$$\lim_{n\to\infty} \sup x_n = \inf_{k\ge 1} \sup_{n\ge k} x_n, \quad \lim_{x\to\infty} \inf x_n = \sup_{k>1} \inf_{n\ge k} x_n. \tag{2.35}$$

You can see the proofs of these attitudes in my book "Quantum Mechanics" [4].

2.3.12 Banach theorem

A given function that maps a metric space into itself, $f : X \to X$. The point $x \in X$ for which $f(x) = x$ is called the *fixed point* of the mapping f. Sufficient conditions for the mapping to have one and only one *fixed point* were given by Banach.

Mapping f of a complete[22] metric space (X, d) into itself is *contraction* if there is a positive number $q < 1$ such that for every pair of points $x_1, x_2 \in X$ the inequality holds

$$d[f(x_1), f(x_2)] \le q d(x_1, x_2). \tag{2.36}$$

[21] Karl Weierstrass (1815-1897), a German mathematician often cited as the "father of modern analysis".

[22] Metric space X is complete if every Cauchy sequence of points in X has a limit that is also in X.

Theorem 2.3.3 (Banach theorem). *The contraction of the f a complete metric space (X,d) into itself has one and only one fixed point.*

Proof. We choose an arbitrary point $x_0 \in X$ and construct a series of successive approximations, points:

$$x-1 = f(x_0), \quad x_2 = f(x_1), \quad \dots, \quad x_n = f(x_{n-1}), \quad \dots \tag{2.37}$$

for $n = 1,2,3,\dots$ and we notice that it is a Cauchy series (whose members with enough large indices are enough close). Really,

$$d(x_{n+1},x_n) = d[f(x_n),f(x_{n-1})] \le qd(x_n,x_{n-1})$$

and repeating this n times we find

$$d(x_{n+1},x_n) \le q^n d(x_1,x_0) = aq^n.$$

Therefore, for $m > n$ is

$$d(x_n,x_m) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{m-1},x_m) \le$$
$$\le aq^n + aq^{n+1} + \dots + aq^{m-1} \le$$
$$\le aq^n + aq^{n+1} + \dots = \frac{aq^n}{1-q} \to 0, \quad m > n \to \infty.$$

Since X is a complete space, there in X exists a point x^* to which the series (x_n) converges, i.e.

$$\lim_{n\to\infty} x_n = x^*. \tag{2.38}$$

But as f is a contraction, it is

$$d[x_{n+1},f(x^*)] = d[f(x_n),f(x^*)] \le qd(x_n,x^*),$$

and so on

$$\lim_{n\to\infty} d[x_{n+1},f(x^*)] = 0,$$

or, what is the same,

$$\lim_{n\to\infty} x_{n+1} = f(x^*).$$

Because (2.38) $x^* = f(x^*)$, i.e. x^* is a fixed point of the mapping f.

This proves the existence of a fixed point, and we further prove that it is the only one. Suppose that f besides x^* has another fixed point $x^{**} \ne x^*$. Then

$$d(x^{**},x^*) = d[f(x^{**}),f(x^*)] \le qd(x^{**},x^*),$$

so because $d(x^{**},x^*) > 0$ follows $q \ge 1$, which is in contradiction with the assumption that f is a contraction. □

We get a simple example of the application of Banach's theorem by putting on the ground a map of the area in which we are. Then there is a single point on the map that exactly matches the place on the ground.

The second example we get starting from the triangle ABC. The midpoints of its sides, the points $A_1 \in BC$, $B_1 \in CA$ and $C_1 \in AB$ form a new triangle $A_1B_1C_1$, and the midpoints of these pages new and so on, n-the midpoints form n-th triangle $A_nB_nC_n$. It can be shown that the sequence of these triangles converges to one point which is the center of gravity of each triangle in the sequence.

A similar example is "picture in picture" (mise en abyme), placing a copy of a picture inside the picture itself. A seemingly infinite series of *recursion* (a procedure or function that uses itself in its definition) is obtained which, according to Banach's theorem, contain a single fixed point.

Example 2.3.4. *Existence of solutions of ordinary equations.*

Explanation. We look for a solution of the equation $F(x) = 0$ of the differentiable function $F(x)$, where $F(a) < 0$, $F(b) > 0$ and $0 < m \le F'(x)) \le 1 - \lambda m$.

Consider an arbitrary real differentiable function $f(x)$ with values in the interval $[a,b]$ such that $|f'(x)| \le q < 1$. Then, based on the mean-value theorem:

$$|f(x_1) - f(x_2)| = |f'(\xi)||x_1 - x_2| \le q|x_1 - x_2|,$$

which due to the definition of the metric on the real line means that f is a contraction and has a fixed point x^* for which $f(x^*) = x^*$.

Put $f(x) = x - \lambda F(x)$, and we will determine the parameter λ later. The given equation is then equivalent to the equation $f(x) = x$, so from $f'(x) = 1 - \lambda F'(x)$ follows

$$1 - \lambda M \le f'(x) \le 1 - \lambda m.$$

It is possible to choose λ so that the constraint on $f'(x)$ is met which justifies the application of Banach's theorem. □

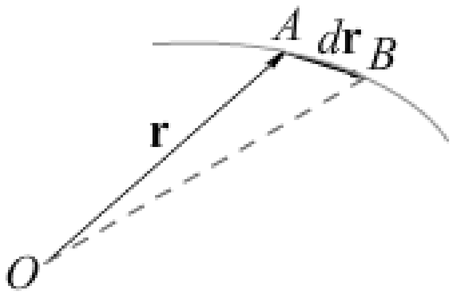
Other important theoretical applications of Banach's theorem are in the proof of the existence of solutions of the system $n \in \mathbb{N}$ of linear algebraic equations, then the proof of the existence of solutions of an infinite system of linear algebraic equations, the existence of a local solution of a first order differential equation which are "regular enough". Also the existence of Fredholm's integral equation[23]. This evidence can be found as examples of the application of Banach's theorem in many textbooks of functional analysis, so I do not list them here.

2.4 Potential

In the figure 2.2, the triangle OAB is spanned by the vectors $\mathbf{r} = \overrightarrow{OA}$ and $d\mathbf{r} = \overrightarrow{AB}$ formed by the infinitesimal displacement $d\mathbf{r}$ of a body on which is acting an (unknown) force from the point O.

The area of the triangle OAB is half the intensity of the vector product of the vectors that spanned it (see [5]), so we can write $d\Pi = \frac{1}{2}\mathbf{r} \times d\mathbf{r}$. It's the first time derivative gives $d\dot{\Pi} = \frac{1}{2}\mathbf{r} \times d\dot{\mathbf{r}}$, and the second $d\ddot{\Pi} = \frac{1}{2}(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}})$. The first item to the right in parentheses is zero, because the vector product of parallel vectors is zero. In the second item, the vector $\ddot{\mathbf{r}}$ is the acceleration of a body that is proportional to the force, so it has the same direction as $\mathbf{r}$. That is why the second item is zero. So, $\ddot{\Pi} = 0$, and hence $\dot{\Pi} = \text{const}$. If the force is gravitational, we have proved *Kepler's second law.* However, the force in that figure may be different.

[23] Fredholm integral equation: https://en.wikipedia.org/wiki/Fredholm_integral_equation



Slika 2.2: Surface of the triangle OAB.

The radius vector from the source of the force O to the body A on which the force acts at equal times erased (whipped) equal surfaces of any kind of force. This will be equally true with the gravitational, electromagnetic, or other attractive or repulsive force of a given dotty source. If this is true, then Newton's law of gravitation cannot be derived from Kepler's other law, that each of the planets of the solar system moves so that its radius vector of position relative to the Sun at equal times passes over equal surfaces.

2.4.1 Newton's law of gravity

We derive Newton's law of gravity from *Kepler's third law*, that the square of the planet's orbital period (T) is proportional to the cubes of their mean distances from the Sun (r).

So, we start from Kepler's proportion $T^2/r^3 = \text{const}$. For circular orbits in general units, it becomes $T^2 = kr^3$, where the orbital period is $T = 2\pi r/v$. Substitute the period to Kepler's proportion and get

$$\frac{4\pi^2 r^2}{v^2} = kr^3,$$

and after arranging and multiplying both sides by mass we find

$$\frac{mv^2}{r} = \frac{4\pi^2 m}{kr^2}.$$

The left side of the equation is the *centripetal force* that keeps the planet in a circular motion. It must be gravity that keeps the body in orbit

$$F = \frac{4\pi^2 m}{kr^2}.$$

According to *Newton's* third law, the force that binds a planet to the Sun is equal to the force that binds the Sun to a planet. Because of this symmetry, action and reaction, we have

$$F = G\frac{Mm}{r^2},$$

where $G = 6,67 \cdot 10^{-11}$ N m^2 kg^{-2} is *gravitational constant*. Force is a vector quantity, so we write this

$$\mathbf{F} = -G\frac{Mm}{r^2}\mathbf{e}_r, \tag{2.39}$$

where the force vector and the unit position vector are written in bold.

In classical mechanics, *work* is the transfer of energy, here W, which is performed by the affecting of a force ($\mathbf{F}$) along a path ($\mathbf{r}$). We say that gravitational and electrostatic forces are *conservative* because the work they do depends only on the endpoints, not on the shape of the road, so

$$dW = \mathbf{F} \cdot d\mathbf{r} = F\,dr\,\cos\varphi, \tag{2.40}$$

where the intensities of the vectors are $F = |\mathbf{F}|$, $r = \mathbf{r}$, and $\varphi = \angle(\mathbf{F}, \mathbf{r})$ is the angle between them.

2.4.2 Moments

Unlike force work, which is a scalar product of the force and path vectors, *force moment* or torque is the vector of the intensity

$$\tau = |\mathbf{r} \times \mathbf{F}| = rF \sin\varphi, \tag{2.41}$$

of the vector product of the path ($\mathbf{r}$) and force ($\mathbf{F}$) vectors. *Angular momentum* ($\mathbf{L}$) is the vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2.42}$$

it is the vector product of the path and line momentum ($\mathbf{p}$) vectors.

In rotational kinematics, the angular momentum takes the place of the force of linear kinematics, because it is formally equivalent to Newton's second law of motion ($F = ma$) where

$$\tau = I\alpha. \tag{2.43}$$

This α becomes angular acceleration, and I rotational inertia. The higher the I, the harder it is to achieve angular acceleration of the body. This inertia explains the figure 2.2 and the constancy of the surfaces at equal times.

On the other hand, we see that the angular momentum (2.42) has a physical dimension of action, that it is a product of path and momentum. In the mentioned picture, the body moves in the field of force so that the action does not change, and now let's add nor the information. We know that information is two-dimensional, that it is the force that can change the probabilities and thus the relative information, but we notice that the force with the source as in the given picture does not do that. It would be different if the force at the source (O) was not constant.

From the point of view of homothety itself, with the increase of r the area of the triangle OAB increases with r^2, and with the same square the area of the sphere (of the radius r with center O) grows. This good synchronization corresponds to the transfer of information from the source to the body using the *virtual spheres* bosons I wrote about earlier, "correcting" the notion of *Feynman diagrams* by aligning it with "information theory".

The amplitude of the "virtual sphere" decreases with the decrease of its surface, and its probability of interaction with another charge decrease too. The wavelength of the "sphere" is constant, so the value of any transmitted momentum is also constant. In short, these are (my) adjustments to the interactions of Feynman diagrams. Here we take on the property of the "sphere" to transmit information and that this information is proportional to the area of a triangle like OAB in the figure 2.2.

Seizing that the area of a triangle is proportional to information, it is also proportional to action, so it is proportional to momentum (and path), i.e. energy (and time). Therefore, the areas that the radius vectors overwrite at equal times represent the energies calculated from (2.40) by integrating

$$W = \int \mathbf{F} \cdot d\mathbf{r}. \tag{2.44}$$

It is energy defined by the scalar multiplication of vectors and is related to the potential. Moments close to it, such as (2.42), are defined by the vector products of the vector. Thus we come to the need for a *pseudoinner product* which I will explain below when we first recall a few examples of potential for completeness.

2.4.3 Potential energy

Gravitational potential, here V, a material point on distant r and mass m is usually defined as the work W required to bring a unit of mass m from infinity to that point:

$$V(r) = \frac{W}{m} = \frac{1}{m}\int_\infty^r \mathbf{F} \cdot d\mathbf{r} = \frac{1}{m}\int_\infty^r G\frac{Mm}{r^2}\,dr = -G\frac{M}{r}. \tag{2.45}$$

The gravitational constant is G, and the gravitational force $\mathbf{F}$ is produced by the mass M. The potential is negative because the force is attractive, which means that the information in the stronger field is smaller. Bodies tend to have less information, but that doesn't mean they succeed. Due to the same inertia, i.e. reluctance to emit information or change action, the body in the gravitational field *falls freely*. It spontaneously remains in the *weightlessness* so moving that its sum of the kinetic:

$$E_k = \int_0^t \mathbf{F} \cdot d\mathbf{r} = \int_0^t \mathbf{v} \cdot d(m\mathbf{v}) = \int_0^v \left(\frac{mv^2}{2}\right) = \frac{mv^2}{2}, \tag{2.46}$$

and the potential energy is always constant. Potential energy is

$$E_p = Fh = mgh, \tag{2.47}$$

where F is the force that would give the mass m the acceleration g on the path h.

Potential energy is the energy that an object has due to its position in relation to another object. We have more potential energy at the top of the stairs than at the bottom, because the earth's gravity is pulling us down. It can do the work by attracting us. When we keep two magnets apart, they have more potential energy than when they are close. If we let them go, the magnets will get closer doing the work.

Electric potential or electric scalar potential is the potential corresponding to an electric field. It is a quantity present at each point in space and is equal to the quotient of potential electricity per unit of charge located in a static (time-invariant) electric field. It is a scalar quantity whose negative *gradient* is equal to the electric field vector

$$\mathbf{E} = -\operatorname{grad}\phi. \tag{2.48}$$

Since the rotor of a stationary electric field is equal to zero, $\nabla \times \mathbf{E} = 0$, this potential (as well as gravitational) does not depend on the path, here C, but only on the endpoints

$$\phi = -\int_C \mathbf{E} \cdot d\mathbf{r}. \tag{2.49}$$

In the case of the charge Q at a distance of r this becomes the *Coulomb*[24] potential

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}, \tag{2.50}$$

where $\varepsilon_0 \approx 8.854 \cdot 10^{-12}$ F m^{-1} is *vacuum permeability.*

In general, the gradient (or gradient of the vector field) of the scalar function $f(\mathbf{x})$ over the vector variable $\mathbf{x} = (x_1, \dots, x_n)$ is denoted by $\operatorname{grad} f$ or *nabla* with the symbol

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right). \tag{2.51}$$

The scalar product of the vectors $\nabla f \cdot \mathbf{v}$, of the gradient at the point $\mathbf{x}$ with the vector $\mathbf{v}$, gives the derivative of the function f at that point in the direction of the given vector.

2.4.4 Pseudoscalar product

We define the *pseudoscalar product* of nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ as a number

$$c = |\mathbf{a}||\mathbf{b}| \sin\angle(\mathbf{a}, \mathbf{b}), \tag{2.52}$$

and we consider it zero if at least one of the factors is zero. It can be found in the Praslov's task collection (see [14]) with the disjunction label, which we change here to $c = \mathbf{a} \wedge \mathbf{b}$ due to later alignment (with Fock space). In my earlier work (see [8]) it is *commutator*[25], and we use both labels here.

We know that expression (2.52) defines the area of a parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ so we can write $\Pi(ABC) = \frac{1}{2}(\overrightarrow{AB} \wedge \overrightarrow{AC})$. In the figure 2.2 the motion is in the plane π with the area of the given triangle $\Pi(OAB) = \frac{1}{2}(\overrightarrow{OA} \wedge \overrightarrow{AB})$.

In general, three points are always in some plane π on which we can place Cartesian rectangular coordinate system OXY. In it are the points $A(A_x, A_y, 0)$, $B(c_x, c_y, 0)$ and $C(C_x, C_y, 0)$ which define the vector product:

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= [(c_x - A_x)\vec{i} + (c_y - A_y)\vec{j}] \times [(C_x - A_x)\vec{i} + (C_y - A_y)\vec{j}] = \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ c_x - A_x & c_y - A_y & 0 \\ C_x - A_x & C_y - A_y & 0 \end{vmatrix} = \vec{k} \begin{vmatrix} c_x - A_x & c_y - A_y \\ C_x - A_x & C_y - A_y \end{vmatrix} \\
&= \vec{k}[(c_x - A_x)(C_y - A_y) - (c_y - A_y)(C_x - A_x)] \\
&= \vec{k}[(A_x c_y - c_x A_y) + (c_x C_y - C_x c_y) + (C_x A_y - A_x C_y)] \\
&= \vec{k}([A, B] + [B, C] + [C, A]),
\end{aligned}$$

where with $[P, Q] = P_x Q_y - Q_x P_y$ we denote the "commutator" of arbitrary points P and Q in the Cartesian system OXY, here in the plane π. Thus

$$\Pi(ABC) = \frac{1}{2}([A, B] + [B, C] + [C, A]) \tag{2.53}$$

is the area of the triangle ABC.

[24] Charles-Augustin de Coulomb (1736-1806), French military engineer and physicist.

[25] I didn't know about the Praslov collection at the time.

Note that $[P,Q] = -[Q,P]$, so for four points in the plane we have the area:

$$\Pi(ABCD) = \frac{1}{2}([A,B] + [B,C] + [C,A]) + \frac{1}{2}([A,C] + [C,D] + [D,A]),$$

so the area of the quadrilateral $ABCD$ written by commutators is

$$\Pi(ABCD) = \frac{1}{2}([A,B] + [B,C] + [C,D] + [D,A]). \tag{2.54}$$

Analogously for the area of an arbitrary polygon $A_1A_2 \dots A_n \subset \pi$ we get

$$2\Pi(A_1A_2 \dots A_n) = [A_1, A_2] + [A_2, A_3] + \dots + [A_{n-1}, A_n]. \tag{2.55}$$

This is an advantage of the commutator method for calculating areas, especially convenient here because the force from a constant source moving the body so that it remains in the same plane.

When one of the themes of the triangle ABC is in the origin, say $C = O(0,0)$ as in the figure 2.2, then the area of the triangle $\Pi(OAB) = \frac{1}{2}[A,B]$. Thus, a commutator is a double area of a triangle by which one side is visible from the opposite vertex.

2.4.5 Multiplication examples

The importance of the pseudoscalar product and thus of the commutator is now greater due to information theory, so it would be good to extract them from naphthalene, if they were once there. Here are some interesting examples.

Example 2.4.1. *For arbitrary vectors* $\mathbf{a} = (A_x, A_y)$, $\mathbf{b} = (c_x, c_y)$ *u* $\mathbf{c} = (C_x, C_y)$ *in the Cartesian plane and an arbitrary number* λ *prove that the relations are valid:*

1. $\mathbf{a} \wedge \mathbf{b} = A_x c_y - A_y c_x$,
2. $(\lambda \mathbf{a}) \wedge \mathbf{b} = \lambda(\mathbf{a} \wedge \mathbf{b})$,
3. $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$.

Proof. 1. Let $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit abscissa and ordinate vectors. Then $\mathbf{e}_x \wedge \mathbf{e}_y = -\mathbf{e}_y \wedge \mathbf{e}_x = 1$, $\mathbf{e}_x \wedge \mathbf{e}_x = \mathbf{e}_y \wedge \mathbf{e}_y = 0$, so $\mathbf{a} \wedge \mathbf{b} = (A_x\mathbf{e}_x + A_y\mathbf{e}_y) \wedge (c_x\mathbf{e}_x + c_y\mathbf{e}_y) = A_x c_y - c_x A_y$.

2. If $\lambda < 0$, then $(\lambda\mathbf{a}) \wedge \mathbf{b} = -\lambda|\mathbf{a}||\mathbf{b}| \sin\angle(-\mathbf{a}, \mathbf{b}) = \lambda|\mathbf{a}||\mathbf{b}| \sin\angle(\mathbf{a}, \mathbf{b}) = \lambda(\mathbf{a} \wedge \mathbf{b})$. For $\lambda \ge 0$ the equation is obvious.

3. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Let the vector $\mathbf{a}$ has orientation of x-axis so $A = (A_x, 0)$, $B = (c_x, c_y)$ and $C = (C_x, C_y)$. Then $\mathbf{a} \wedge \mathbf{b} = A_x c_y$, $\mathbf{a} \wedge \mathbf{c} = A_x C_y$, so $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = A_x(c_y + C_y) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$. □

Example 2.4.2. *For arbitrary points A, B, C and D prove the areas:*

1. $\Pi(ABC) = -\Pi(BAC) = \Pi(BCA)$,
2. $\Pi(ABC) = \Pi(DAB) + \Pi(DBC) + \Pi(DCA)$.

Proof. 1. Follows from $\overrightarrow{AB} \wedge \overrightarrow{AC} = \overrightarrow{AB} \wedge (\overrightarrow{AB} + \overrightarrow{BC}) = -\overrightarrow{BA} \wedge \overrightarrow{BC} = \overrightarrow{BC} \wedge \overrightarrow{BA}$.

2. Derived in order from:

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = (\overrightarrow{AD} + \overrightarrow{DB}) \wedge (\overrightarrow{AD} + \overrightarrow{DC}) =$$

$$= \overrightarrow{AD} \wedge \overrightarrow{DC} + \overrightarrow{DB} \wedge \overrightarrow{AD} + \overrightarrow{DB} \wedge \overrightarrow{DC}$$
$$= \overrightarrow{DC} \wedge \overrightarrow{DA} + \overrightarrow{DA} \wedge \overrightarrow{DB} + \overrightarrow{DB} \wedge \overrightarrow{DC}.$$

□

Example 2.4.3. *Three runners A, B and C run along parallel tracks at constant speeds. At the initial moment the area of triangle ABC was 2, and after 5 seconds it was 3. What could it be after another 5 seconds?*

Solution. At start, $t = 0$, we have $\overrightarrow{AB} = \mathbf{v}$ и $\overrightarrow{AC} = \mathbf{w}$. Then in time $t > 0$ we have $\overrightarrow{AB} = \mathbf{v} + t(\mathbf{b} - \mathbf{a})$ and $\overrightarrow{AC} = \mathbf{w} + t(\mathbf{c} - \mathbf{a})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are velocities of runners A, B and C respectively. How velocity vectors are parallel to being $(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) = 0$, so $|\Pi(ABC)| = \frac{1}{2}|\overrightarrow{AB} \wedge \overrightarrow{AC}| = |x + ty|$,where x and y are some constants.

The conditions give the system $|x| = 2$, $|x + 5y| = 3$, so for $t = 10$ we get two solutions, 4 or 8, for the required areas. □

The next examples would be various calculations of the motion of the planets of the solar system, then in other fields of forces, and then something similar in information theory.

2.4.6 Generalization

The commutator method in a way derives from the known noncommutativity of especially linear physics operators, so it is natural to return to it. It would be consistent to say that the circular product of three commutators is equal to the double *triangle area* they form, whatever that means in terms of classical geometry. Let's look at the example of *Pauli matrices*:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.56}$$

where $i^2 = -1$ holds for the imaginary unit. For the corresponding "triangle area" we get:

$$2\hat{\Pi}(\hat{\sigma}_x\, \hat{\sigma}_y\, \hat{\sigma}_z) = [\hat{\sigma}_x, \hat{\sigma}_y] + [\hat{\sigma}_y, \hat{\sigma}_z] + [\hat{\sigma}_z, \hat{\sigma}_x] =$$
$$= (\hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x) + (\hat{\sigma}_y\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_y) + (\hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z)$$
$$= 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + 2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so:

$$\hat{\Pi}(\hat{\sigma}_x\, \hat{\sigma}_y\, \hat{\sigma}_z) = \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}, \quad \det\hat{\Pi} = 3. \tag{2.57}$$

On the right is the determinant of the matrix $\hat{\Pi}(\hat{\sigma}_x\, \hat{\sigma}_y\, \hat{\sigma}_z)$, the surface of Pauli's let's say triangle.

We get similar for *quaternions* (see [7]), here:

$$\hat{q}_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \hat{q}_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{q}_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.58}$$

It is now the double "triangle area":

$$2\hat{\Pi}(\hat{q}_x\, \hat{q}_y\, \hat{q}_z) = [\hat{q}_x, \hat{q}_y] + [\hat{q}_y, \hat{q}_z] + [\hat{q}_z, \hat{q}_x] =$$

$$= (\hat{q}_x\hat{q}_y - \hat{q}_y\hat{q}_x) + (\hat{q}_y\hat{q}_z - \hat{q}_z\hat{q}_y) + (\hat{q}_z\hat{q}_x - \hat{q}_x\hat{q}_z)$$

$$= 2\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + 2\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so

$$\hat{\Pi}(\hat{q}_x\,\hat{q}_y\,\hat{q}_z) = \begin{pmatrix} -i & -1-i \\ 1-i & -i \end{pmatrix}, \quad \det\hat{\Pi} = 1. \tag{2.59}$$

The determinant of the oriented "quaternion area" of this order is one third of Pauli's; in a different order of circumnavigation of the "vertex" it would be equal to Pauli's.

Another example would be the linear operators $\hat{A}$ and $\hat{B}$ with *eigenvector* $\mathbf{v}$ of the *commutator*:

$$[\hat{A}, \hat{B}]\mathbf{v} = (A_x c_y - A_y c_x)\mathbf{v}, \tag{2.60}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. We ask that the following relations also apply:

$$\hat{A}\mathbf{u} = A_x\mathbf{v}, \quad \hat{A}\mathbf{v} = A_y\mathbf{u}, \quad \hat{B}\mathbf{u} = c_x\mathbf{v}, \quad \hat{B}\mathbf{v} = c_y\mathbf{u}. \tag{2.61}$$

These scalars A_x, A_y, c_x, c_y are (anti) eigenvalues associated with *anti-eigenvectors* of $\mathbf{u}$ and $\mathbf{v}$ given operators $\hat{A}$ and $\hat{B}$.

That equation (2.60) really holds follows from (2.61) in order:

$$[\hat{A}, \hat{B}]\mathbf{v} = (\hat{A}\hat{B} - \hat{B}\hat{A})\mathbf{v} = \hat{A}\hat{B}\mathbf{v} - \hat{B}\hat{A}\mathbf{v} = \hat{A}(c_y\mathbf{u}) - \hat{B}(A_y\mathbf{u}) =$$

$$= (\hat{A}\mathbf{u})c_y - (\hat{B}\mathbf{u})A_y = A_x c_y\mathbf{v} - c_x A_y\mathbf{v} = (A_x c_y - c_x A_y)\mathbf{v},$$

which is true. Similarly, using (2.61), we check:

$$[\hat{A}, \hat{B}]\mathbf{u} = -(A_x c_y - A_y c_x)\mathbf{u}, \tag{2.62}$$

hence the name "anti-eigen" for the vectors $\mathbf{u}$ and $\mathbf{v}$.

Example 2.4.4. *Find the eigenvectors of the Pauli matrix commutator (2.56).*

Solution. We calculate the eigenvectors of the commutator, in order:

$$[\hat{\sigma}_x, \hat{\sigma}_y]\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad [\hat{\sigma}_y, \hat{\sigma}_z]\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad [\hat{\sigma}_z, \hat{\sigma}_x]\mathbf{v}_3 = \lambda_3\mathbf{v}_3,$$

$$2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad 2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \lambda_3\begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

$$2i\begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} = \lambda_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad 2i\begin{pmatrix} y_2 \\ x_2 \end{pmatrix} = \lambda_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad 2\begin{pmatrix} y_3 \\ -x_3 \end{pmatrix} = \lambda_3\begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We have normalized these vectors. All three corresponding eigenvalues are equal to each other, $\lambda_1 = \lambda_2 = \lambda_3 = 2i$. Note that the commutators given in the upper order are equal to the quaternions $-2\hat{k}$, $2\hat{j}$ and $-2\hat{i}$. □

Example 2.4.5. *Find "anti-eigen" vectors of Pauli matrices (2.56).*

Solution. We start from the first commutator from the previous example and (2.61), where $\mathbf{v} = \mathbf{v}_1$, $\hat{A} = \hat{\sigma}_x$, $\hat{B} = \hat{\sigma}_y$, and the scalars A_x, A_y, c_x, c_y and the vectors $\mathbf{u} = \binom{x}{y}$ are unknown:

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = A_x\begin{pmatrix}1\\ 0\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = A_y\begin{pmatrix}x\\ y\end{pmatrix},\ \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = c_x\begin{pmatrix}1\\ 0\end{pmatrix},\ \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = c_y\begin{pmatrix}x\\ y\end{pmatrix},$$

$$\begin{pmatrix}y\\ x\end{pmatrix} = A_x\begin{pmatrix}1\\ 0\end{pmatrix},\quad \begin{pmatrix}0\\ 1\end{pmatrix} = A_y\begin{pmatrix}x\\ y\end{pmatrix},\quad \begin{pmatrix}-iy\\ ix\end{pmatrix} = c_x\begin{pmatrix}1\\ 0\end{pmatrix},\quad \begin{pmatrix}0\\ i\end{pmatrix} = c_y\begin{pmatrix}x\\ y\end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix}0\\ 1\end{pmatrix},\quad A_x = 1,\quad A_y = 1,\quad c_x = -i,\quad c_y = i.$$

Indeed, the eigenvalue of the first commutator is $A_x c_y - c_x A_y = 2i$.

For the second commutator ($\mathbf{v} = \mathbf{v}_2$) we find similarly:

$$\hat{\sigma}_y\mathbf{u} = A_y\mathbf{v},\quad \hat{\sigma}_y\mathbf{v} = A_z\mathbf{u},\quad \hat{\sigma}_z\mathbf{u} = c_y\mathbf{v},\quad \hat{\sigma}_z\mathbf{v} = c_z\mathbf{u},$$

$$\begin{pmatrix}-iy\\ ix\end{pmatrix} = A_y\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 1\end{pmatrix},\quad \frac{\sqrt{2}}{2}\begin{pmatrix}-i\\ i\end{pmatrix} = A_z\begin{pmatrix}x\\ y\end{pmatrix},\quad \begin{pmatrix}x\\ -y\end{pmatrix} = c_y\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 1\end{pmatrix},\quad \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ -1\end{pmatrix} = c_z\begin{pmatrix}x\\ y\end{pmatrix},$$

$$\mathbf{u} = \frac{\sqrt{2}}{2}\begin{pmatrix}-i\\ i\end{pmatrix},\quad A_y = 1,\quad A_z = 1,\quad c_y = -i,\quad c_z = i.$$

Indeed, the eigenvalue of the second commutator is $A_y c_z - c_y A_z = 2i$.

For the third commutator ($\mathbf{v} = \mathbf{v}_3$) we find:

$$\hat{\sigma}_z\mathbf{u} = A_z\mathbf{v},\quad \hat{\sigma}_z\mathbf{v} = A_x\mathbf{u},\quad \hat{\sigma}_x\mathbf{u} = c_z\mathbf{v},\quad \hat{\sigma}_x\mathbf{v} = c_x\mathbf{u},$$

$$\hat{\sigma}_z\begin{pmatrix}x\\ y\end{pmatrix} = A_z\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix},\quad \hat{\sigma}_z\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix} = A_x\begin{pmatrix}x\\ y\end{pmatrix},\quad \hat{\sigma}_x\begin{pmatrix}x\\ y\end{pmatrix} = c_z\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix},\quad \hat{\sigma}_x\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix} = c_x\begin{pmatrix}x\\ y\end{pmatrix},$$

$$\begin{pmatrix}x\\ -y\end{pmatrix} = A_z\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix},\quad \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ -i\end{pmatrix} = A_x\begin{pmatrix}x\\ y\end{pmatrix},\quad \begin{pmatrix}y\\ x\end{pmatrix} = c_z\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ i\end{pmatrix},\quad \frac{\sqrt{2}}{2}\begin{pmatrix}i\\ 1\end{pmatrix} = c_x\begin{pmatrix}x\\ y\end{pmatrix},$$

and from there:

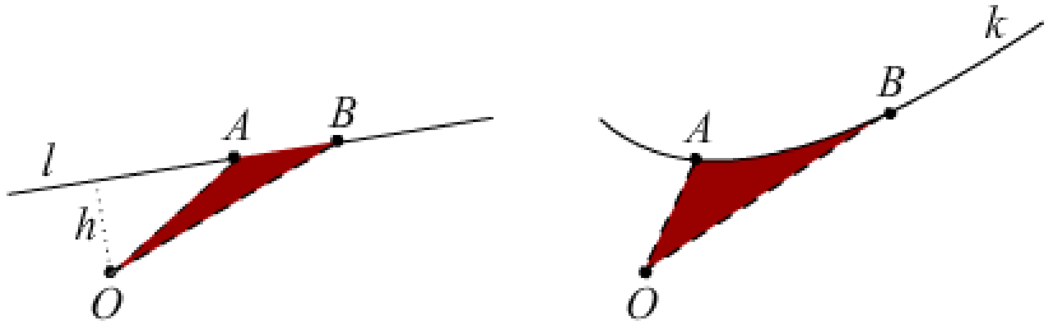
$$\mathbf{u} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ -i\end{pmatrix},\quad A_z = 1,\quad A_x = 1,\quad c_z = -i,\quad c_x = i.$$

Indeed, the eigenvalue of the third commutator is $A_z c_x - c_z A_x = 2i$. □

2.4.7 Potential information

If we understand what is meant by the "area of the triangle of operators", then we can go back to the *Kepler* second law to connect it with Heisenberg's *relations of uncertainty* and the principle of least action. Let's start with quite simple examples in the following picture.

In the figure 2.3 on the left, we see the point O and at a distance h from it a straight line l containing two points A and B. If we move (translate) AB along the given line without changing its length, then the area of triangle OAB does not change either. This is Kepler's second law when there is no force in the O source. When there is some constant force in the source O in the figure 2.3 on the right, then we have the situation we considered in the figure 2.2, except that now the force is repulsive.



Slika 2.3: Surfaces of the triangles OAB.

The source is a charge, a fermion that emits virtual bosons in the form of concentric spheres whose radii increase over time. It sends virtual information to space about the shortest distances that are not straight lines, that is, that the paths of least action are no longer rectilinear for other particles of appropriate charge.

Under the action of force, space becomes like a curved surface (saddle, surface sphere, or some third) for an ant whose rectilinear movement is hindered by a solid surface, but which the ant still crosses "the shortest paths", realizing it in its own way. We are in the same situation when we cross a hill by the shortest path which is not a road through a hill, if there are no tunnels.

Such an effect is produced by a force changing the probabilities, making earlier rectilinear trajectories less probable, which means more informative, with greater consumption of action, which the physical body will now avoid by adhering to the principle of least action.

Theoretically, we can take shorter and shorter lengths of the AB trajectory k tending to the infinitesimal boundary with always the same statement, that the pull OA passing to OB in the same time intervals defines constantly equal surfaces, but practically it will end on Planck's order constants. We finally return to the commutator method of unitary operators of quantum mechanics.

In the figure 2.3 on the right, the hatched area OAB is still half the value of the commutator $[A, B]$, but the points A and B are then the operators of position and momentum of the particle, and the commutator and "constant surface" of the order of magnitude of the quantum of action.

Thus generalized the Kepler's second law speaks of motion in a field where there is a source of constant force, or outside it, according to the principle of least action. That was the topic of my previous book, "Minimalism of Information" [2] in a part that I would not repeat. What I am emphasizing here is the movement due to the change of potential.

Potential energy is part of the action (product of energy and duration) of space itself. It is thus part of the obstacles and passages for the particles that react to it, what in the theory of relativity is called *geometry of space*, and what we could call here the space of physical probabilities, or physical information.

Bodies move along, from their point of view, the most probable trajectories, which are the paths of spatial minimum information. This stems from the general, principled minimalism of information emission. This principle is so general that it also applies to the bodies themselves. They move through such differently informative space at such variable speeds that their total information remains constant. But unlike classical materialist physics, information is not characterized by the communication of everything with everything, and then not even the uniformity of charge.

2.5 Combining eigenvalues

We consider the eigenvalues of the commutators and the "anti-eigen" values of their individual vector space operators X over the scalar body Φ. Due to their specific role in the commutator, I also call these "pseudo-eigenvalues", or *combined-eigenvalues*. It is assumed that the operators are linear and with the usual notation $\hat{A}, \hat{B}, \hat{C}, \dots$, unlike the vectors they act on here in the notation $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$, unless otherwise stated.

Let the linear operators $\hat{A}$, $\hat{B}$ and the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be given such that:

$$\hat{A}\mathbf{x} = A_y\mathbf{y}, \quad \hat{B}\mathbf{y} = c_x\mathbf{x}, \quad \hat{B}\mathbf{x} = c_y\mathbf{z}, \quad \hat{A}\mathbf{z} = A_x\mathbf{x}, \tag{2.63}$$

where A_x, c_x, A_y, c_y are scalars. We can assume that these vectors are unit numbers, the operators are unitary, and the scalars are complex numbers, but this is not important for most statements and exceptions will be especially emphasized. The commutator of the operator $\hat{A}$ and $\hat{B}$ is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \tag{2.64}$$

so is its eigen (characteristic) equality

$$[\hat{A}, \hat{B}]\mathbf{x} = (A_x c_y - A_y c_x)\mathbf{x}, \tag{2.65}$$

which follows from (2.63). The expression $\hat{T}\mathbf{x} = \lambda\mathbf{x}$ has an eigenvector and a corresponding eigenvalue. Several such eigenvalues make up the "spectrum" of the operator $\hat{T}$.

2.5.1 Operator spectrum

The scalar $\lambda \in \Phi$ is called the characteristic, or eigenvalue of the operator $\hat{T} \in (X \to X)$, if there is at least one vector $\mathbf{x} \in X$, $x \neq 0$, so $\hat{T}\mathbf{x} = \lambda\mathbf{x}$. This vector is called the characteristic, eigenvector of the operator $\hat{T}$, and the set of all eigenvectors of a given operator is called the *spectrum* labeled $\lambda(\hat{T})$, or $\sigma(\hat{T})$.

Operators form a vector space, so that in the case of a finite dimensional space in an infinite sequence of degrees of the operator:

$$\hat{I}, \quad \hat{T}, \quad \hat{T}^2, \quad \hat{T}^3, \quad \dots, \tag{2.66}$$

has the most natural number of $n \in \mathbb{N}$ linearly independent vectors. This means that there is a number $m \in \mathbb{N}$ that we can write

$$\hat{T}^m = a_1\hat{T}^{m-1} + \dots + a_{m-1}\hat{T} + a_m\hat{I}, \tag{2.67}$$

for some scalars $a_1, \dots, a_m \in \Phi$. When the powers of the operators are linearly independent vectors, then the coefficients a_k are uniquely determined. Then it's a polynomial

$$p(\lambda) = \lambda^m - a_1\lambda^{m-1} - \dots - a_{m-1}\lambda - a_m \tag{2.68}$$

uniquely determined by the operator $\hat{T}$. Now equation (2.67) takes the simple form $p(\hat{T}) = 0$, and we call polynomial (2.68) the *minimal polynomial* of the given operator.

The minimality of this polynomial follows from the view that the m operators in the sequence (2.66) are linearly independent vectors. Namely, if $p(t)$ were a polynomial of degree

q less than m and if were $p(\hat{A}) = 0$, then the vectors $\hat{I}, \hat{A}, \dots, \hat{A}^q$ were linearly dependent, which is a contradiction.

The second thing we see from (2.67) is the equality

$$\frac{1}{\alpha_m}(\hat{T}^{m-1} - \alpha_1\hat{T}^{m-2} - \dots - \alpha_{m-1}\hat{I})\hat{T} = \hat{I}, \tag{2.69}$$

when $\alpha_m \neq 0$, that is

$$\hat{T}^{-1} = \frac{1}{\alpha_m}(\hat{T}^{m-1} - \alpha_1\hat{T}^{m-2} - \dots - \alpha_{m-1}\hat{I}) \tag{2.70}$$

reciprocal, or the *inverse operator* of the $\hat{T}$ operator that exists when $\alpha_m \neq 0$. That the inverse operator is commutative with a given operator follows from this very definition, because $\hat{T}$ commutes with $\hat{T}^k$. An operator for which there is an inverse operator is called *regular* or invertible, and if such does not exist for the operator, we say that it is *singular*.

The basic properties of the $\mathcal{G}$ group of regular operators are obvious. Namely, the inverse to the inverse operator is the operator $(\hat{T}^{-1})^{-1} = \hat{T}$, so from $\hat{T} \in \mathcal{G}$ followed by $\hat{T}^{-1} \in \mathcal{G}$. If two operators are part of that group then it is also their composition $\hat{T}_1, \hat{T}_2 \in \mathcal{G} \implies \hat{T}_1 \circ \hat{T}_2 \in \mathcal{G}$, because $\hat{T}_2^{-1}\hat{T}_1^{-1}(\hat{T}_1\hat{T}_2) = \hat{T}_2^{-1}(\hat{T}_1^{-1}\hat{T}_1)\hat{T}_2 = \hat{I}$. With $\hat{T}$ in the group are all the powers $\hat{T}^k$ for every $k \in \mathbb{Z}$. In particular, the *bijection* (two-sided unique mapping) of a group to a group, $f : \mathcal{G} \to \mathcal{G}'$, is called *isomorphism* if $f(g_1 \circ g_2) = f(g_1) \circ f(g_2)$.

The set $\mathcal{G}$ of all regular operators of a finite dimensional vector space is a group. If $\lambda \in \Phi$ is the zero of the minimal polynomial $p(\lambda)$ of the operator $\hat{T}$, then $\lambda\hat{I} - \hat{T}$ is a singular operator. The set solution of the minimal polynomial is called the spectrum, in other words if $p(\lambda) = 0$ then $\lambda \in \sigma(\hat{T})$. If $\lambda \in \Phi$ is not the zero of the minimal polynomial of the operator $\hat{T}$, then $\lambda\hat{I} - \hat{T}$ is a regular operator and its inverse value is called *resolvent*

$$R_\lambda = (\lambda\hat{I} - \hat{T})^{-1} = \frac{1}{\mu(\lambda)}(\hat{T}_0\lambda^{m-1} + \hat{T}_1\lambda^{m-2} + \dots + \hat{T}_{m-2}\lambda + \hat{T}_{m-1}). \tag{2.71}$$

Here is

$$\hat{T}_k = \hat{T}^k - a_1\hat{T}^{k-1} - \dots - a_{k-1}\hat{T} - a_k\hat{I}, \quad \hat{T}_0 = \hat{I}, \tag{2.72}$$

for $k = 1, \dots, m-1$, marked from (2.67). In addition, for every λ is

$$\hat{T}_0\lambda^{m-1} + \hat{T}_1\lambda^{m-2} + \dots + \hat{T}_{m-2}\lambda + \hat{T}_{m-1} \neq 0. \tag{2.73}$$

These are important well-known views that you can find more about in linear algebra textbooks, in English for example [13].

Cayley–Hamilton theorem, which is perhaps the most significant statement in this area of linear algebra[26], says that each quadratic *matrix* cancels out its characteristic polynomial. For example, the matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}, \tag{2.74}$$

has the characteristic polynomial

$$p(\lambda) = \det(\lambda\hat{\mathbf{I}} - \hat{\mathbf{A}}) = \det(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}) = \det\begin{pmatrix} \lambda - 1 & -4 \\ -3 & \lambda - 2 \end{pmatrix} =$$

[26] `http://web.mit.edu/2.151/www/Handouts/CayleyHamilton.pdf`

$$= \begin{vmatrix} \lambda - 1 & -4 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - (-3)(-4) = \lambda^2 - 3\lambda - 10.$$

The spectrum of this matrix, $\sigma(\hat{\mathbf{A}})$, consists of two eigenvalues:

$$\lambda_{1,2} = \frac{3 \pm \sqrt{3^2 - 4 \cdot (-10)}}{2} = \frac{3 \pm 7}{2} = \begin{cases} 5 \\ -2 \end{cases},$$

the solutions of the quadratic equation $\lambda^2 - 3\lambda - 10 = 0$.

We define this minimal polynomial by an (arbitrary) matrix $\hat{\mathbf{X}}$ of type 2×2 with

$$p(\hat{\mathbf{X}}) = \hat{\mathbf{X}}^2 - 5\hat{\mathbf{X}} - 2\hat{\mathbf{I}}, \tag{2.75}$$

so:

$$p(\hat{\mathbf{A}}) = \hat{\mathbf{A}}^2 - 3\hat{\mathbf{A}} - 10\hat{\mathbf{I}} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}^2 - 3\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 1 + 4 \cdot 3 & 1 \cdot 4 + 4 \cdot 2 \\ 3 \cdot 1 + 2 \cdot 3 & 3 \cdot 4 + 2 \cdot 2 \end{pmatrix} - \begin{pmatrix} 3 \cdot 1 & 3 \cdot 4 \\ 3 \cdot 3 & 3 \cdot 2 \end{pmatrix} - \begin{pmatrix} 10 \cdot 1 & 10 \cdot 0 \\ 10 \cdot 0 & 10 \cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 \\ 9 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 12 \\ 9 & 6 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \hat{\mathbf{0}}.$$

Indeed, the square matrix (2.74) nullifies its characteristic polynomial (2.75).

Example 2.5.1. *Find the characteristic polynomial of the matrix*

$$\hat{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.76}$$

determine its spectrum and test the Cayley–Hamilton theorem.

Solution. We find the characteristic polynomial:

$$p(\lambda) = \det(\lambda\hat{\mathbf{I}} - \hat{\mathbf{A}}) = \det\begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix} =$$

$$= (\lambda - a_{11})(\lambda - a_{22}) - a_{21}a_{12} = \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21})$$

$$= \lambda^2 - \lambda \operatorname{tr} \hat{\mathbf{A}} + \det \hat{\mathbf{A}}.$$

The solutions of the corresponding quadratic equation are the spectrum of the given matrix, and checking the characteristic polynomial is not a difficult task either. □

The space of unitary operators is to the isomorphism equal to the corresponding space of matrices, and both to *evolutions* of quantum mechanics. As we know, *quantum mechanics* is a representation of Hilbert's algebra whose operators make a vector space dual to that on which they act. Eigenvectors are *quantum states*, and eigenvalues that are real numbers are *observables*.

2.5.2 Eigenvectors in steps

When $\lambda \in \Phi$ is an *eigenvalue* of the (linear) operator $\hat{T}$, then the set of vectors $\mathbf{x} \in X$ with property $\hat{T}\mathbf{x} = \lambda\mathbf{x}$ forms a subspace we call the *proper subspace* of a given operator associated with a given value.

Namely, from $\hat{T}(\alpha'\mathbf{x}' + \alpha''\mathbf{x}'') = \alpha'\hat{T}\mathbf{x}' + \alpha''\hat{T}\mathbf{x}'' = \alpha'\lambda\mathbf{x}' + \alpha''\lambda\mathbf{x}'' = \lambda(\alpha'\mathbf{x}' + \alpha''\mathbf{x}'')$ it follows that any vector that is a linear combination of eigenvalues, $\mathbf{x} = \alpha'\mathbf{x}' + \alpha''\mathbf{x}''$, is also an eigenvector of the same operator, $\hat{T}\mathbf{x} = \lambda\mathbf{x}$. The action of the operator $\hat{T}$ in that subspace is reduced to the multiplication of a vector by a scalar, then always an eigenvector with an eigenvalue.

In the interpretation of quantum mechanics, evolution (the operator $\hat{T}$) translates the inherent quantum states (vector $\mathbf{x}$) into similar, say electrons into electrons, atom into atom, while the eigenvalue is (the scalar λ) observed, such as energy, position, number of particles. However, evolutions can be cascading, in steps.

In the commutator structure (2.64) is the sum for which $\hat{A}\hat{B}\mathbf{x} = A_x c_x \mathbf{x}$ holds. This is also a characteristic (auxiliary) equation of the form $\hat{T}\mathbf{x} = \lambda\mathbf{x}$, where $\hat{T} = \hat{A}\hat{B}$ is the composition of the operators, and the eigenvalue is $\lambda = A_x c_y$. It is written in more detail:

$$\hat{A}\hat{B}\mathbf{x} = \hat{A}\hat{B}(\alpha'\mathbf{x}' + \alpha''\mathbf{x}'') = \hat{A}(\alpha'\hat{B}\mathbf{x}' + \alpha''\hat{B}\mathbf{x}'') = \hat{A}(\alpha' B_y'\mathbf{y}' + \alpha'' B_y''\mathbf{y}'') =$$

$$= \hat{A}(\frac{\alpha' B_y'}{c_y}\mathbf{y}' + \frac{\alpha'' B_y''}{c_y}\mathbf{y}'')c_y = \hat{A}(\beta'\mathbf{y}' + \beta''\mathbf{y}'')c_y = \hat{A}\mathbf{y}c_y = A_x c_y \hat{x}.$$

So, we have the composition of inherent mapping:

$$\hat{B} : \mathbf{x} \to c_y \mathbf{y}, \quad \hat{A} : \mathbf{y} \to A_x \mathbf{x}, \tag{2.77}$$

where are the vectors $\mathbf{x} = \alpha'\mathbf{x}' + \alpha''\mathbf{x}''$ and $\mathbf{y} = \beta'\mathbf{y}' + \beta''\mathbf{y}''$.

For example, for the characteristic polynomial $p_\sigma(\lambda) = \det(\lambda\hat{\mathbf{I}} - \hat{\sigma})$ of each of the three Pauli matrices (2.56) we get

$$p_\sigma(\lambda) = \lambda^2 - 1, \tag{2.78}$$

with the spectrum of real numbers $\lambda_{1,2} = \pm 1$, the solutions of the equation $p(\lambda) = 0$. The characteristic polynomial $p_q(\lambda) = \det(\lambda\hat{\mathbf{I}} - \hat{q})$ of each of the three quaternions (2.58) is

$$p_q(\lambda) = \lambda^2 + 1, \tag{2.79}$$

with a spectrum of imaginary numbers $\lambda_{1,2} = \pm i$. It is:

$$\begin{cases} \hat{\sigma}_x\hat{\sigma}_y = \hat{q}_z, & \hat{\sigma}_y\hat{\sigma}_z = \hat{q}_x, & \hat{\sigma}_z\hat{\sigma}_x = \hat{q}_y, \\ \hat{q}_x\hat{q}_y = -i\hat{\sigma}_z, & \hat{q}_y\hat{q}_z = -i\hat{\sigma}_x, & \hat{q}_z\hat{q}_x = -i\hat{\sigma}_y, \end{cases} \tag{2.80}$$

which means that Pauli matrices can be factorized into quaternions, and quaternions into Pauli matrices.

Example 2.5.2. *Find the eigenvectors of Pauli matrices and quaternions.*

Solution. We solve equations $\hat{\sigma}_k\mathbf{w}_k = \lambda\mathbf{w}_k$, for indexes $k \in \{x, y, z\}$, each for two eigenvalues, $\lambda = \pm 1$, and we normalize the eigenvectors, $\|\mathbf{w}_k\| = 1$. We get:

$$\begin{cases} \mathbf{w}_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \mathbf{w}_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, & \mathbf{w}_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \mathbf{w}_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, & \mathbf{w}_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, & \mathbf{w}_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{cases} \tag{2.81}$$

These are the eigenvectors of the Pauli matrices (2.56). We then solve the characteristic quaternion equation $\hat{q}_k \mathbf{w}_k = \lambda \mathbf{w}_k$, for the indices $k \in \{x, y, z\}$, each for two eigenvalues, $\lambda = \pm i$ and eigenvectors are also normalized, $\|\mathbf{w}_k\| = 1$. We obtain identical corresponding values (2.81). The eigenvectors of quaternions and Pauli matrices are equal, but their eigenvalues are not equal. □

Note that the last sentence of the solution says that the same quantum states (eigenvectors) do not have to be equally observable. It is a confirmation of what we are constantly working with in information theory, that active and passive information are equal in quantity but not in properties.

The eigenvalue of the unit matrix $\hat{\mathbf{I}}$ is one, and its eigenvector is each vector. It, like the unit operator $\hat{I}$, represents identical transformations, *status quo*, a quantum evolution that does not change anything. However, the square of each of the Pauli matrices is a unit matrix:

$$\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_y^2 = \hat{\mathbf{I}}. \tag{2.82}$$

The same is true for the corresponding operators that make space isomorphic to matrices. Hence, the status quo can be considered as a cascading transformation of non-identical ones, by the composition of two using any pair of the three Pauli's, and then further, according to (2.80), each Pauli matrix is decomposed into two quaternion factors.

Example 2.5.3. *Divide the Pauli matrices into two quaternion factors and present (2.77), the processes of eigenvector transformation.*

Solution. In the case of the first Pauli matrix $\hat{\sigma}_x = i\hat{q}_y\hat{q}_z$ we have:

$$\begin{cases} \hat{\sigma}_x \mathbf{w}_{x+} = \hat{q}_y(i\hat{q}_z\mathbf{w}_{x+}) = \hat{q}_y[\frac{i}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}] = \hat{q}_y\mathbf{w}_{x-} = \mathbf{w}_{x+}, \\ \hat{\sigma}_x \mathbf{w}_{x-} = \hat{q}_y(i\hat{q}_z\mathbf{w}_{x-}) = \hat{q}_y[\frac{i}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}] = \hat{q}_y\mathbf{w}_{x+} = -\mathbf{w}_{x-}, \end{cases} \tag{2.83}$$

In the case of the second Pauli matrix $\hat{\sigma}_y = i\hat{q}_z\hat{q}_x$ it is:

$$\begin{cases} \hat{\sigma}_y \mathbf{w}_{y+} = \hat{q}_z(i\hat{q}_x\mathbf{w}_{y+}) = \hat{q}_z[\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}] = -\hat{q}_z\mathbf{w}_{y-} = \mathbf{u}_{y+}, \\ \hat{\sigma}_y \mathbf{w}_{y-} = \hat{q}_z(i\hat{q}_x\mathbf{w}_{y-}) = \hat{q}_z[\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix}] = -\hat{q}_z\mathbf{w}_{y+} = -\mathbf{w}_{y-}, \end{cases} \tag{2.84}$$

In the case of the third Pauli matrix $\hat{\sigma}_z = i\hat{q}_x\hat{q}_y$ we calculate:

$$\begin{cases} \hat{\sigma}_z \mathbf{w}_{z+} = i\hat{q}_x(\hat{q}_y\mathbf{w}_{z+}) = i\hat{q}_x[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = -i\hat{q}_x\mathbf{u}_{z-} = \mathbf{w}_{z+}, \\ \hat{\sigma}_z \mathbf{w}_{z-} = i\hat{q}_x(\hat{q}_y\mathbf{w}_{z-}) = i\hat{q}_x[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}] = i\hat{q}_y\mathbf{w}_{z+} = -\mathbf{w}_{z-}, \end{cases} \tag{2.85}$$

□

Individual Pauli matrices (as well as quaternions) have one eigenvector (2.81), each with its own direction forming one-dimensional eigen-subspaces. If such a matrix is represented by the other two (2.80) then the composition of the mapping of the eigenvector (2.77) takes place in cascade through different subspaces. Analogously, quantum evolution has its periodic phases that do not have to be all real in terms of classical quantum mechanics.

2.5.3 Example of linear systems

We observe a system of two linear equations with two unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = y_1 \\ a_{21}x_1 + a_{22}x_2 = y_2 \end{cases}, \tag{2.86}$$

which we write $\hat{\mathbf{A}}\mathbf{x} = \mathbf{y}$ for short. We found the characteristic polynomial of this matrix in the example 2.5.1, where it can be seen that zero and its trace form its spectrum, *iff* (if and only if) its determinant is zero. On the other hand, the determinant is not zero if a given system of equations has a unique solution.

This can be used to further discuss the example 2.5.3. Let's say that in addition to the above, the following system of equations is given

$$\begin{cases} b_{11}z_1 + b_{12}z_2 = x_1 \\ b_{21}z_1 + b_{22}z_2 = x_2 \end{cases}, \tag{2.87}$$

which we write briefly $\hat{\mathbf{B}}\mathbf{z} = \mathbf{x}$. By composing these systems we get $\hat{\mathbf{A}}\hat{\mathbf{B}}\mathbf{z} = \mathbf{y}$, and we write this matrix $\hat{\mathbf{C}}\mathbf{z} = \mathbf{y}$, or in the form of a system of linear equations

$$\begin{cases} c_{11}z_1 + c_{12}z_2 = x_1 \\ c_{21}z_1 + c_{22}z_2 = x_2 \end{cases}, \tag{2.88}$$

where we have the composition of linear mappings as a product of matrices $\hat{\mathbf{C}} = \hat{\mathbf{A}}\hat{\mathbf{B}}$. It is known that the determinant of the product is equal to the product of the determinants, $\det\hat{\mathbf{A}}\hat{\mathbf{B}} = \det\hat{\mathbf{A}}\det\hat{\mathbf{B}}$, and we can now use that to check the regularity of the compositions of such systems.

For example, consider the following three matrices:

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}, \quad \hat{\mathbf{B}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \hat{\mathbf{C}} = \begin{pmatrix} 9 & 6 \\ 7 & 8 \end{pmatrix}. \tag{2.89}$$

The product of the first two is the third. The determinants are $\det\hat{\mathbf{A}} = -10$, $\det\hat{\mathbf{B}} = -3$, and $\det\hat{\mathbf{C}} = 30$. Their spectra are the sets $\sigma(\hat{\mathbf{A}}) = \{5, -2\}$, $\sigma(\hat{\mathbf{B}}) = \{3, -1\}$ and $\sigma(\hat{\mathbf{C}}) = \{15, 2\}$, say solutions of $\lambda_{1,2}$ characteristic equations

$$\lambda^2 - \lambda \operatorname{tr}\hat{\mathbf{M}} + \det\hat{\mathbf{M}} = 0, \tag{2.90}$$

where we take $\hat{\mathbf{M}} \in \{\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\}$. Based on Vieta's formulas, it is further

$$\lambda_1 + \lambda_2 = \operatorname{tr}\hat{\mathbf{M}}, \quad \lambda_1\lambda_2 = \det\hat{\mathbf{M}}, \tag{2.91}$$

which can also be used to check such results, but also to draw a simple common matrix with the same eigenvalues.

For a given eigenvalue λ we define the eigenvector as a solution of the matrix equation $\hat{\mathbf{M}}\mathbf{m}_\lambda = \lambda\mathbf{m}_\lambda$, so we take only the unit vectors:

$$\begin{cases} \mathbf{a}_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \mathbf{b}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \mathbf{c}_{15} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \mathbf{a}_{-2} = \frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}, & \mathbf{b}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, & \mathbf{c}_2 = \frac{1}{\sqrt{85}}\begin{pmatrix} 6 \\ -7 \end{pmatrix}. \end{cases} \tag{2.92}$$

There are no significant changes if we move to unit matrices $\hat{\mathbf{M}} \to \hat{\mathbf{M}}/\det\hat{\mathbf{M}}$.

Example 2.5.4. *Consider the phases of the vector* $\mathbf{c}_{15}$ *through the factors of the matrix* $\hat{\mathbf{C}}$.

Solution. We calculate the steps:

$$\hat{\mathbf{C}}\mathbf{c}_{15} = 15\mathbf{c}_{15} \longrightarrow \begin{pmatrix} 9 & 6 \\ 7 & 8 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{15}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\hat{\mathbf{A}}\hat{\mathbf{B}}\mathbf{c}_{15} = 15\mathbf{c}_{15} \longrightarrow \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{15}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\hat{\mathbf{A}}3\mathbf{b}_3 = 15\mathbf{c}_{15} \longrightarrow \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \frac{3}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{15}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$15\mathbf{a}_5 = 15\mathbf{c}_{15} \longrightarrow \frac{15}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{15}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In this cascading process, the initial eigenvector does not change direction. □

The inherent subspaces in this case were the same for all three matrices, so analogously in quantum processes we conclude that such phases do not have to change the nature of the quantum state. However, the following example shows that there is another possibility.

Example 2.5.5. *Describe the stages of transformation of the vector* $\mathbf{c}_2$ *through the process* $\hat{\mathbf{C}}$.

Solution. I write clearly:

$$\hat{\mathbf{C}}\mathbf{c}_2 = 2\mathbf{c}_2 \longrightarrow \begin{pmatrix} 9 & 6 \\ 7 & 8 \end{pmatrix} \frac{1}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix} = \frac{2}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix},$$

$$\hat{\mathbf{A}}\hat{\mathbf{B}}\mathbf{c}_2 = 2\mathbf{c}_2 \longrightarrow \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \frac{1}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix} = \frac{2}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix},$$

$$\hat{\mathbf{A}} \frac{1}{\sqrt{85}} \begin{pmatrix} -8 \\ 5 \end{pmatrix} = 2\mathbf{c}_2 \longrightarrow \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \frac{1}{\sqrt{85}} \begin{pmatrix} -8 \\ 5 \end{pmatrix} = \frac{2}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix},$$

$$\frac{1}{\sqrt{85}} \begin{pmatrix} 12 \\ -14 \end{pmatrix} = 2\mathbf{c}_2 \longrightarrow \frac{1}{\sqrt{85}} \begin{pmatrix} 12 \\ -14 \end{pmatrix} = \frac{2}{\sqrt{85}} \begin{pmatrix} 6 \\ -7 \end{pmatrix}.$$

In this cascade process, the initial eigenvector changes direction. □

The eigenvector $\mathbf{c}_2$ replaces the direction $\binom{6}{-7}$ with $\binom{-8}{5}$. By analogy with quantum processes, it is a kind of change in the nature of the quantum state. We saw an even more drastic change in the example 2.5.3, when one phase is imaginary.

2.5.4 Quantum mechanics operators

Each measurable parameter in a physical system is represented by a quantum-mechanical *operator*. Such operators arise when we describe the nature of quantum mechanics by means of waves (wave functions), and not of discrete particles, especially not those whose motions and dynamics are reduced to determined equations of Newtonian physics. Values such as coordinates and components of velocity, line and torque of particles and functions of these quantities, formerly variable classical mechanics, are now described by linear operators [27].

[27] ScienceDirect: `https://www.sciencedirect.com/`

For example, the operator $\hat{Q} : f \to g$ acts on the function $f = f(x)$ and transforms it into another function $g = g(x)$. The result of the action of that operator on the same function f can be the product of the constant (scalar) λ and that function

$$\hat{Q} : f \to \lambda f, \tag{2.93}$$

when we say that the λ is *eigenvalue* and function f *eigenfunction* of the operator $\hat{Q}$. The relation $\hat{Q}f = \lambda f$ is called *eigen equation*.

An example of an eigen equation is *Schrödinger*[28] time-independent equation

$$\hat{H}\psi(x) = E\psi(x), \tag{2.94}$$

along the abscissa (x-axis), where $\psi(x)$ is *wave function*, and

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \hat{V} \tag{2.95}$$

a linear total energy operator called Hamiltonian by *Hamilton*[29], $\hbar = h/2\pi = 1.054$ Js is *reduced* Planck's constant, m is the mass of the quantum system (particles), and $\hat{V} = V(x)$ is the potential operator of type (2.93).

When the Schrödinger time-independent equation is applied to three spatial dimensions (length, latitude and altitude) in Cartesian rectangular coordinates is

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{2.96}$$

Here ∇ *nabla* is an operator or scalar field gradient, $\Delta = \nabla^2$ is the *Laplacian*

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{2.97}$$

or *Laplace*[30] operator, and $\mathbf{r} = (x, y, z)$ position vector. The expression in parentheses is a spatial, time-independent Hamiltonian in the form (2.94) and the substitution $x \to \mathbf{r}$.

The time-dependent spatial Schrödinger equation is

$$\hat{H}\psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t), \tag{2.98}$$

where $\hat{H}$ is the spatial Hamiltonian. It reduces to the time-dependent Schrödinger equation of abscissa by substitution $\mathbf{r} \to x$.

The position operator is a multiplication by position, along the abscissa or spatially, in order:

$$\hat{x} = x, \quad \hat{\mathbf{r}} = \mathbf{r}. \tag{2.99}$$

These are formally operators such as potential, or in general (2.93).

The line momentum operator of a particle moving along the abscissa is

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}. \tag{2.100}$$

[28] Erwin Schrödinger (1887-1961), Austrian physicist.
[29] William Rowan Hamilton (1805-1865), Irish mathematician.
[30] Pierre-Simon Laplace (1749-1827), French mathematician.

Its spatial form is

$$\hat{\mathbf{p}} = -i\hbar\nabla. \tag{2.101}$$

Note that the square of this operator divided by the mass represents the kinetic energy, and when we add the potential to that energy we get the Hamiltonian.

The kinetic energy operator is:

$$E_k = \frac{p_x^2}{2m}, \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}, \quad -\frac{\hbar^2}{2m}\nabla^2, \tag{2.102}$$

respectively in classical and quantum physics along the abscissa and spatial.

The operator of the rotational momentum is

$$\hat{\mathbf{L}} = -i\hbar(\hat{\mathbf{r}} \times \nabla). \tag{2.103}$$

In short, it is a vector product of the position and line momentum operators. Note that the components of this vector are:

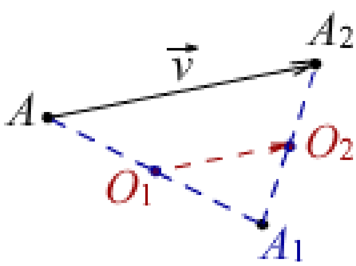
$$\begin{cases} \hat{L}_x = y\hat{p}_z - z\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \\ \hat{L}_y = z\hat{p}_x - x\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \\ \hat{L}_z = x\hat{p}_y - y\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \end{cases} \tag{2.104}$$

Compare this with the commutators and pseudo-scalar products discussed here.

2.5.5 Rotations

According to *Noether's theorem*[31], with each *conservation law* there is a symmetry that is *isometry* (preserves distances), and consistently further, symmetry consists of *rotations*, so the study of rotations becomes very important for quantum mechanics.

For example, *mirror symmetry* is the rotation for the straight line angle (90°) around the plane of the mirror (in the new dimension), *axial symmetry* is the rotation for the extended angle around the axis in space. *Central Symmetry* is the rotation about a point of symmetry for the straight line angle in any plane of the line containing the original and the image of the point. *Translation* for the vector $\vec{v} = 2 \cdot \overrightarrow{O_1O_2}$ is obtained using two central symmetries, which can be seen in Figure 2.4.



Slika 2.4: Translation from two rotations.

Namely, the central symmetry $s_1 : A \to A_1$ maps the point A to the point A_1 around the center O_1 so that the three points are on the same line $A - O_1 - A_1$ and that the distances of

[31] `https://www.mathpages.com/home/kmath564/kmath564.htm`

the original and the copy from the center is unchanged $O_1A = O_1A_1$. Analogously, the central symmetry $s_2 : A_1 \to A_2$ maps the point A_1 to the point A_2 around the center O_2. Thus we get the triangle AA_1A_2 with the midline O_1O_2 which, as is known from geometry, is parallel to the base AA_2 and equal to half of its length. Therefore, the vector $\vec{v} = \overrightarrow{AA_2} = 2 \cdot \overrightarrow{O_1O_2}$, as said.

Let us now rotate in Cartesian rectangular system OXY the point $A(x,y)$ around the origin O for the angle $\omega = \angle(AOA')$ to the point $A'(x',y')$. The double area of the triangle OAA' is equal to the commutator, and on the other hand it is equal to the product of the lengths of the two sides and the sine of the angle between them:

$$2\Pi(\Delta OAA') = [A, A'] = xy' - x'y = |OA||OA'|\sin\omega. \tag{2.105}$$

The cosine of the angle of rotation can be obtained from the scalar product, by definition, and multiplying the coordinates:

$$\overrightarrow{OA} \cdot \overrightarrow{OA'} = |OA||OA'|\cos\omega = xx' + yy'. \tag{2.106}$$

Since $|OA| = |OA'|$ and $|OA||OA'| = x^2 + y^2$, we find from the previous:

$$x\cos\omega - y\sin\omega = \frac{x^2x' + xyy'}{x^2 + y^2} - \frac{xyy' - x'y^2}{x^2 + y^2} = \frac{x^2x' + x'y^2}{x^2 + y^2} = x',$$

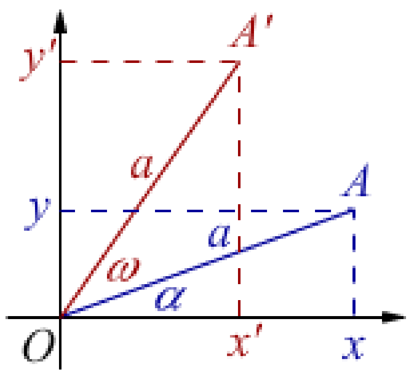
$$x\sin\omega + y\cos\omega = \frac{x^2y' - xx'y}{x^2 + y^2} + \frac{xx'y + y^2y'}{x^2 + y^2} = \frac{x^2y' + y^2y'}{x^2 + y^2} = y'.$$

From there

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.107}$$

It is the transformation of the point $A(x,y)$ into the point $A'(x',y')$ by rotating for the angle ω around the origin of the coordinate system OXY. The matrix in this equation, of type 2×2 with variable angle ω, defines the group $SO(2)$.

Formula (2.107) can be obtained with the "school" image 2.5. The points $A(x,y)$ and $A'(x',y')$ are in the same system OXY and form equal lengths a to the origin, with an angle between $\omega = \angle(AOA')$ and the abscissa angle to closer $\alpha = \angle(XOA)$.



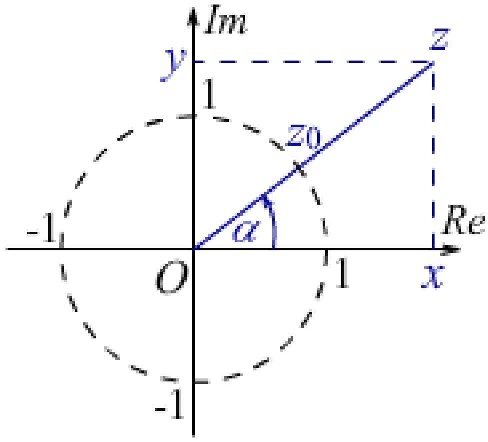
Slika 2.5: Rotation "scholastic".

From the image 2.5 we read:

$$\begin{cases} x' = a\cos(\alpha + \omega) = a(\cos\alpha\cos\omega - \sin\alpha\sin\omega) = x\cos\omega - y\sin\omega, \\ y' = a\sin(\alpha + \omega) = a(\cos\alpha\sin\omega + \sin\alpha\cos\omega) = x\sin\omega + y\cos\omega, \end{cases} \tag{2.108}$$

and that come down to the previous.

In the following figure, 2.6 is a plane of complex numbers $z = x + iy$, where the square of the imaginary unit $i^2 = -1$. The point z has coordinates (x, y) with projections on the abscissa and ordinate in the order $\Re(z) = x$ and $\Im(z) = y$, real and imaginary part. *Modulo* is the length $|z| = Oz = \sqrt{x^2 + y^2}$, and *argument* the oriented angle from abscissa to number, $\alpha = \angle(xOz)$.



Slika 2.6: The plane of complex numbers $\mathbb{C}$.

Conjugated to the complex number $z = x + iy$ is the complex number $z^* = x - iy$. Their product is the real number $z^* z = x^2 + y^2 = |z|^2$, so $z_0^* z_0 = 1$. From the right triangles we find $x = |z| \cos\alpha$ and $y = |z| \sin\alpha$ and form

$$z = |z|(\cos\alpha + i\sin\alpha). \tag{2.109}$$

This form gives special significance to the points on the (intermittently drawn) circle of a unit radius with the center at the origin. In the figure 2.6 such a point is $z_0 = \operatorname{cis}\alpha$

$$\operatorname{cis}\alpha = \cos\alpha + i\sin\alpha, \tag{2.110}$$

because its modulo $|z_0| = 1$. The product of that unit complex number and an arbitrary complex number

$$w = u + iv = |w|(\cos\beta + i\sin\beta) \tag{2.111}$$

increases the argument of the number w by α:

$$w' = u' + iv' = |w|(\cos\beta + i\sin\beta)(\cos\alpha + i\sin\alpha) =$$

$$= |w|[(\cos\beta\cos\alpha - \sin\beta\sin\alpha) + i(\cos\beta\sin\alpha + \sin\beta\cos\alpha)]$$

$$= |w|[\cos(\beta + \alpha) + i\sin(\beta + \alpha)],$$

and that is rotation. Multiplying the complex number of arguments β by the complex number of arguments α gives the complex number of arguments $\beta + \alpha$ and the modulus equal to the product of the module of these two numbers

$$w \cdot z = |w|(\cos\beta + i\sin\beta) \cdot |z|(\cos\alpha + i\sin\alpha) = |w||z|[\cos(\beta + \alpha) + i\sin(\beta + \alpha)]. \tag{2.112}$$

Multiplication by a unit complex number ($\operatorname{cis}\omega$) is equivalent to multiplication by the matrix (2.107) of the group $SO(2)$.

The group $SO(2)$ is a peculiarly, two-dimensional case of *special orthogonal group* $SO(n)$ consisting of *orthogonal matrices*, those whose determinants are one. This group, which is also called the *rotation group*, because it generalizes rotations in space of dimension $n = 2$ or $n = 3$, is widely studied today.

In general, the orthogonal group $O(n)$ of dimension n, which we sometimes call *general orthogonal group* in analogy with the "general linear group" $GL(n)$, is the group *isometry* (transformation which preserves the distance) of Euclidean space of dimension n and has a fixed point, where the operation of the group is compositions of the transformations. The orthogonal matrix has real coefficients, and its *inverse* (multiplied by the matrix gives a unit matrix) is equal to its *transposed* (matrix whose columns are replaced by rows, and rows by columns). The orthogonal group is an algebraic group and *Lie group* – organized with continuous and smooth elements, as opposed to a discrete group.

The special unitary group $SU(n)$ of degree n, is the Lie group of unitary matrices $n \times n$ determinant one. In the more general case, unitary matrices can have a complex determinant of absolute value one. Group operation is matrix multiplication. A special unitary group is a subgroup of the *unitary group* $U(n)$ consisting of all $n \times n$ matrices, and this subgroup of the general linear group $GL(n)$. The group $SU(n)$ has wide application in the *standard model* of particle physics, especially $SU(2)$ in *electroweak* interactions, and $SU(3)$ in quantum *chromodynamics*.

However, here we will not go beyond the results of my contribution [9], for example, from the expression of an arbitrary matrix 2×2 using Pauli matrices, or quaternions:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2}\hat{\mathbf{I}} + \frac{a_{12} + a_{21}}{2}\hat{\sigma}_x + i\frac{a_{12} - a_{21}}{2}\hat{\sigma}_y + \frac{a_{11} - a_{22}}{2}\hat{\sigma}_z, \tag{2.113}$$

where $\hat{\mathbf{I}}$ is the corresponding unit matrix to Pauli (2.56). We easily get and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2}\hat{\mathbf{I}} - i\frac{a_{12} + a_{21}}{2}\hat{q}_x + \frac{a_{12} - a_{21}}{2}\hat{q}_y - i\frac{a_{11} - a_{22}}{2}\hat{q}_z, \tag{2.114}$$

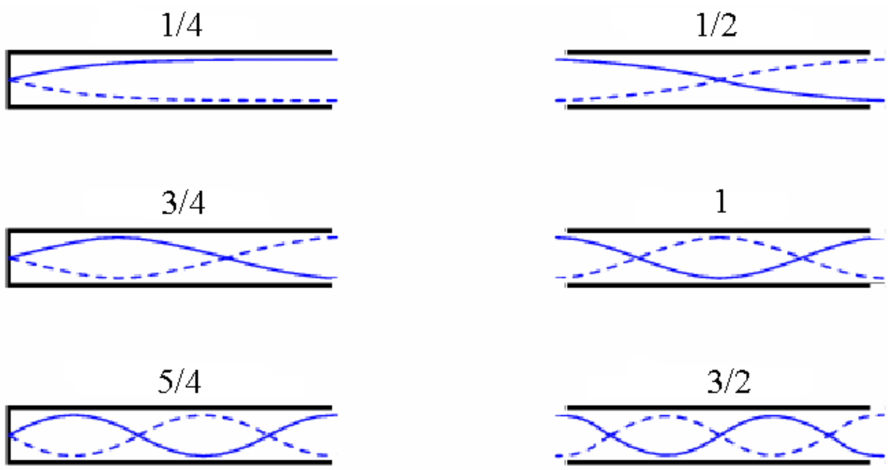
for quaternions (2.58).

2.6 Examples of eigenvalues

In physics problems given by partial differential equations, we usually look for eigenvalues using boundary conditions. The models of such are *sound waves* through the tubes of aerial musical instruments.

The flute is a tube whose active length is changed by opening and closing holes. The effective length of the trombone is changed by sliding the tube in or out. The organ uses tubes of different fixed lengths, each with a different base frequency. In each of these examples, the sound produced by the tube is determined by its length L.

In the figure 2.7 the pipes are open on one and both sides. One *wavelength* λ is one period of a sinusoid and we only see it in the middle picture on the right. These are open pipes for which, on the right, above and below, $L = \frac{1}{2}\lambda$ and $L = \frac{3}{2}\lambda$ are mentioned in order. In the pictures on the left are half-open pipes and from top to bottom is $L = \frac{1}{4}\lambda$, $L = \frac{3}{4}\lambda$ and $L = \frac{5}{4}\lambda$.



Slika 2.7: Number of waves on the length L of pipes.

We get these and similar forms of pressure by solving the *wave equation* which describes the sound along the abscissa

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{2.115}$$

where P is *acoustic pressure* (local deviation from ambient pressure), and $v = \lambda f$ is *speed of sound* (in air under normal conditions about 340 meters per second), $f = 1/T$ is *frequency*, and T is the period. Assuming that the speed of sound is constant, the general solution is

$$P = P_1(vt - x) + P_2(vt + x), \tag{2.116}$$

where P_1 and P_2 are twice differentiable functions. The solution can be understood as *superposition* of two waves of arbitrary profile, the first traveling in the direction of the x-axis and the second in the opposite direction, both at the speed of v. Some solutions are sinusoids

$$P = P_0 \sin(\omega t \pm kx), \tag{2.117}$$

where P_0 is a constant, $\omega = 2\pi f$ is a circular frequency, and $k = 2\pi/\lambda$ is a wave number.

2.6.1 Standing waves

Consider further only waves that do not move, *standing waves* in a tube open on both sides. At the pipe boundaries $x = 0$ and $x = L$ is the normal atmospheric pressure, there is no change in it and developing the sine of the sum (difference) according to the addition formulas of functions like (2.117) we find

$$P(x,t) = \psi(x)\cos(\omega t + \phi_0), \tag{2.118}$$

where ϕ_0 is a constant, a phase shift, and $\psi(x)$ is a time-independent function. Equation (2.115) thus becomes a stationary wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + k^2\psi = 0, \quad k = \frac{\omega}{v}, \tag{2.119}$$

where k could also be a wave vector.

This is a typical example of the *characteristic equation.* The general solution to this problem of eigenvalues, as we know, is

$$\psi(x) = C_1 \sin(kx) + C_2 \cos(kx), \tag{2.120}$$

where C_1 and C_2 are arbitrary constants. Since $\cos(kx)$ does not satisfy the boundary condition, that there is no pressure change at the point $x = 0$, the general solution (2.119) with the given boundary conditions is reduced to

$$\psi(x) = C \sin(kx), \tag{2.121}$$

with an arbitrary constant C. It describes a standing wave. The second initial condition, when $x = L$ then $\sin(kL) = 0$, gives more detailed eigenvalues of the wave vector

$$k = k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots \tag{2.122}$$

and hence wavelengths

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \dots \tag{2.123}$$

where in the case of $n = 1$ we have *base wave*, and for $n = 2, 3, \dots$ we have *harmonics* or higher tones. The corresponding frequencies are

$$f_n = \frac{nv}{2L}, \quad n = 1, 2, 3, \dots, \tag{2.124}$$

because $f = \omega/(2\pi) = v/\lambda$.

Therefore, the general solution of wave equation (2.118) for changes in acoustic pressure inside a pipe open on both sides is the superposition of all these solutions

$$P(x,t) = \sum_{n=1}^{\infty} C_n \sin(k_n x) \cos(\omega_n t + \phi_n), \quad k_n = \frac{\omega_n}{v} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots, \tag{2.125}$$

where the coefficients C_n and the phase shifts ϕ_n are arbitrary. Pipes closed at both ends (flutes) have a similar solution, and the solutions of semi-open pipes are slightly different.

Eigenfunctions that correspond to different eigenvalues are mutually *orthogonal*, because

$$\int_0^L \sin(k_m x) \sin(k_n x)\, dx = \frac{L}{2}\delta_{mn}. \tag{2.126}$$

Normalized eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(k_n x), \tag{2.127}$$

of course, with the previous property of orthogonality

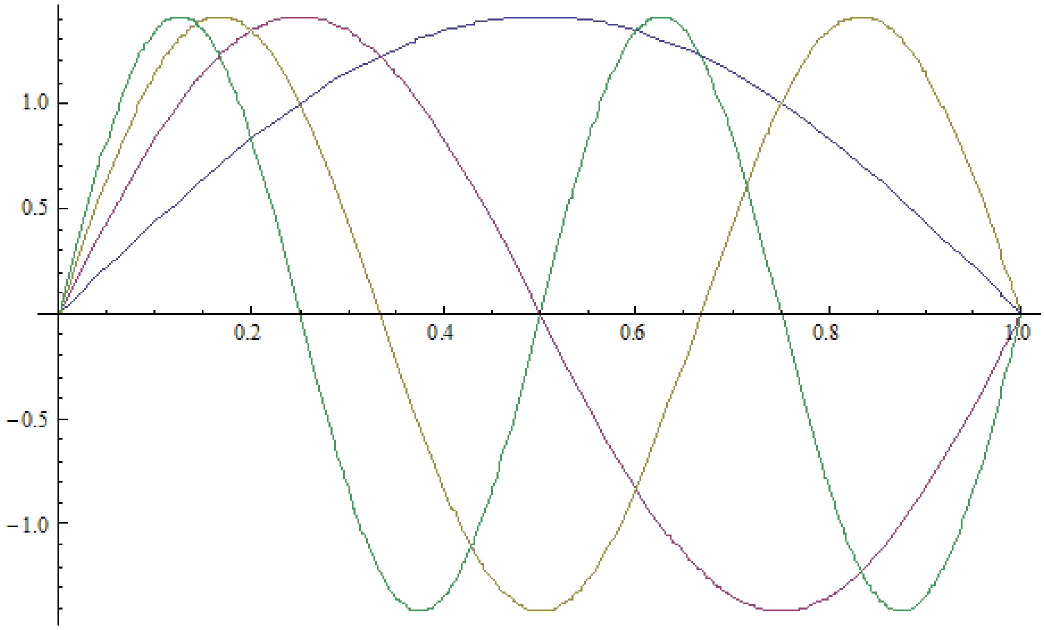
$$\int_0^L \psi_m(x)\psi_n \, dx = \delta_{mn}, \tag{2.128}$$

where $\delta_{mn} = 1$ when $m = n$, and $\delta_{mn} = 0$ when $m \ne n$.

The spatial formulation of this problem is

$$\Delta\psi(\vec{r}) = \lambda\psi(\vec{r}), \tag{2.129}$$

where Δ is Laplacian (2.97), and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector. An example is Schrödinger's equation (2.98).



Slika 2.8: Standing waves of pipes $L = 1$, for $n = 1, 2, 3, 4$.

2.6.2 Particle in a box

Calculating the wave of a particle moving along an abscissa interval from which it cannot escape is the basic school task of quantum mechanics. Solutions to this problem are possible values of its energy E and wave function $\psi(x)$ whose squares of intensity represent the probabilities of the position of the particle at the place x with the given energy.

We solve this task in four steps. We define the potential energy V, we solve the Schrödinger equation, we look for a solution for the wave function, and finally for the allowed energies.

A box is an interval $0 < x < L$ within the potential $V = 0$ and which grows to infinity at the ends ($V = \infty$ for $x < 0$ or $x > L$). The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \tag{2.130}$$

where $\hbar$ is the reduced Planck constant, m is the mass of the particle, $\psi(x)$ stationary time-independent wave function, $V(x)$ potential energy as a function of position, E energy, real number.

Note that this case is easily reduced to the motion of a free particle along the x-axis with zero energy potential ($V = 0$ everywhere) with the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x), \tag{2.131}$$

with the general solution (2.120). Boundary conditions and differentiation again give:

$$\psi(x) = C\sin(kx),$$

$$\frac{d\psi}{dx} = kC\cos(kx),$$

$$\frac{d^2\psi}{dx^2} = -k^2 C\sin(kx),$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

as we have already seen in the case of the open air tube with sound.

According to Schrödinger's equation, the wave number is now

$$k = \left(\frac{8\pi^2 mE}{h^2}\right)^{1/2}, \tag{2.132}$$

where $h = 6.626\times 10^{-34}$ is *Planck's constant*, so the wave function is

$$\psi(x) = C\sin\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} x, \tag{2.133}$$

where we need to find the value of the constant C.

Boundary conditions say that the probability of finding a particle in places $x = 0$ and $x = L$ is zero. When $x = L$ will be:

$$C\sin\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} L = 0,$$

$$\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} L = n\pi, \quad n = 1,2,3,\dots$$

and hence

$$\psi(x) = C\sin\frac{n\pi}{L}x, \tag{2.134}$$

which is in accordance with (2.122).

The constant C will give the condition of normalizing the probability, that the probability of the particle in the box is one, i.e. that it is not somewhere outside the box:

$$\int_0^L \psi^2\,dx = 1,$$

$$C^2\int_0^L \sin^2\left(\frac{n\pi}{L}\right)x\,dx = 1,$$

$$C = \sqrt{\frac{2}{L}},$$

and thence

$$\psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x, \quad n = 1,2,3,\dots \tag{2.135}$$

which agrees with (2,127).

Substituting this into Schrödinger's equation (2.131) we find the allowable particle energies in the box

$$E_n = \frac{n^2h^2}{8\pi L^2}, \quad n = 1,2,3,\dots, \tag{2.136}$$

which means that its energies are quantized, and that the lowest possible energy of the particle is not zero. A particle can never be at complete rest, it must always have some kinetic energy, which is also in accordance with Heisenberg's relations and the *uncertainty principle*.

Wondering where the complex numbers are here? They "celebrated" the calculations of quantum mechanics, so I'm demonstrating the same *particle in a box* problem in another way.

The general solution of equation (2.131) is actually

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{2.137}$$

where A and B are arbitrary constants, and $k = \pm\sqrt{2mE}/\hbar$ is the wave number. How it is

$$e^{i\alpha} = \cos\alpha + i\sin\alpha = \operatorname{cis}\alpha,$$

the function (2.110), that is:

$$\psi(x) = A[\cos(kx) + i\sin(kx)] + B[\cos(kx) - i\sin(kx)] =$$
$$= (A+B)\cos(kx) + i(A-B)\sin(kx),$$

so putting $A + B = C_2$ and $A - B = -iC_1$ we get (2.120).

These are well-known substitutions in solving differential equations, but now we have the opportunity to notice their connection with Heisenberg's relations of uncertainty, with action and information in the way I treat them, that is, with the logarithm of probability that is such generalized information ($I = -\log p$). Precisely because quantum mechanics works with complex numbers, when logarithms are periodic functions, its particles are also waves, they have a probabilistic wave nature.

More detailed solutions of "particle in a box" can be found in the books of Quantum Mechanics, my [4] or something similar. There where Schrödinger's equation is specifically solved for the so-called potential step, potential and high threshold, potential and deep pit, then potential well, and further for various particle penetrations through obstacles, all different cases of boundary conditions. One of these cases is the tunnel effect, which I will explain briefly here.

2.6.3 Tunnel effect

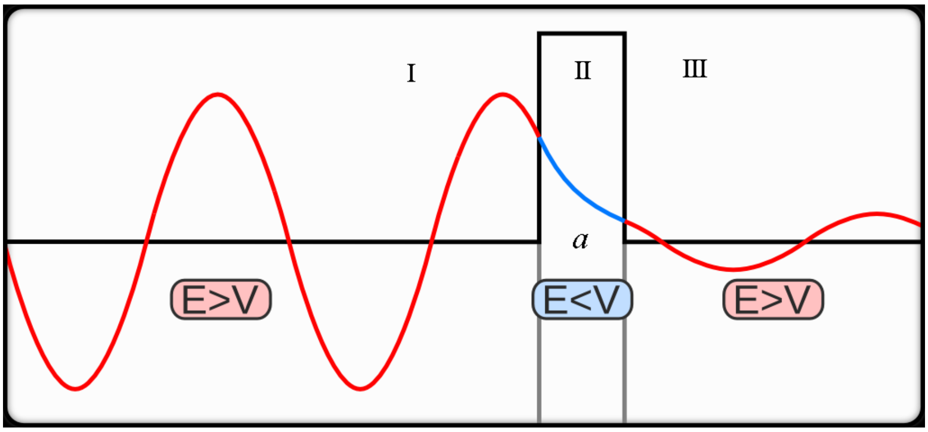
Tunneling or *tunnel effect* in physics is the penetration of particles through high energy obstacles. According to classical physics the tunneling would be impossible, but not in quantum.

The tunnel effect was first noticed by Robert Williams Wood in 1897 as an electron penetration, but he failed to interpret it. Researchers of radioactive decay (1899) expressed the possibility that the decay occurred due to the tunnel effect, but it was theoretically described only by George Gamow (1929) after Rutherford and co-workers discovered that the alpha particle is actually a helium nucleus. Today, however, the discovery of the tunnel effect is attributed to Friedrich Hund, who (1926-27) considered it when measuring isomerism in molecules.

When a particle of mass m and energy E encounters an obstacle of energy V, if $E < V$ it cannot pass it according to classical physics, because its kinetic energy E_k would then be negative and the momentum p imaginary size

$$E_k = \frac{p^2}{2m} = E - V. \tag{2.138}$$

In quantum mechanics, in which the probabilities of finding a particle behind an obstacle are calculated from the Schrödinger equation, such a passage is possible with the same energy of the E particle after penetration, but with a lower probability of finding it.



Slika 2.9: Tunneling along the abscissa.

We see this in the figure 2.9 (Wikipedia, Quantum tunneling) of a particle extending along the abscissa (x-axis) with amplitudes on the ordinate (y-axis). Wavelengths indicate the energy of a particle, and amplitudes indicate the probability of finding it at a given place. Again we solve the 1-D Schrödinger equation with constant potential

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0, \tag{2.139}$$

whose general solution is

$$\psi = A\exp[i\frac{x}{\hbar}\sqrt{2m(E-V)}] + B\exp[-i\frac{x}{\hbar}\sqrt{2m(E-V)}]. \tag{2.140}$$

There are three areas of the particle in the figure: before obstacle I, at obstacle II, and behind obstacle III. In the first and third domains, the potential energy is zero $V = 0$ and then the particle energy is greater than the potential ($E > 0$), but in the second domain the potential energy is greater than the particle energy ($E < V$). Therefore, we distinguish three solutions:

$$\begin{cases} \psi_1 = A_1\exp(ikx) + c_1\exp(-ikx), & x \in I, \\ \psi_2 = A_2\exp(-\chi x) + c_2\exp(\chi x), & x \in II, \\ \psi_3 = A_3\exp[ik(x-a)] + c_3\exp[-ik(x-a)], & x \in III, \end{cases} \tag{2.141}$$

where it is assumed that the barrier of width $a > 0$. In addition, $c_3 = 0$, because that member describes a repulsed wave moving from infinity backwards, which does not exist here.

In quantum mechanics, the probability density of particles of mass m in the quantum state $\psi(\mathbf{r}, t)$ is defined by

$$\rho = \psi^*\psi = |\psi|^2, \tag{2.142}$$

so the probability P of finding a particle in the infinitesimal volume element $d^3\mathbf{r}$ is

$$dP = |\psi|^2 d^3\mathbf{r}. \tag{2.143}$$

The number of particles, a vector that passes vertically through a unit area in a unit of time, is called *probability flux* and is

$$\mathbf{J} = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi), \tag{2.144}$$

and it is also called "probability current", or "current density", or "flux density probability". In our case, the probability flux is a scalar

$$J = \frac{i\hbar}{2m}(\frac{\partial\psi^*}{\partial x}\psi - \frac{\partial\psi}{\partial x}\psi^*). \tag{2.145}$$

The barrier permeability coefficient D is the ratio of the flow density of particles that have passed and the flow densities of those that have arrived.

We assume that χa is large enough, so we calculate:

$$A_2 = \frac{1-in}{2}A_3\exp(\chi a), \quad c_2 = \frac{1+in}{2}A_3\exp(-\chi a) \approx 0,$$

$$A_1 = \frac{(1-in)(1+i/n)}{4}A_3\exp(\chi a),$$

substitutions are also in use:

$$n = \frac{k}{\chi} = \sqrt{\frac{E}{V-E}}, \quad D_0 = \frac{16n^2}{(1+n^2)^2}.$$

This leads to an obstacle permeability coefficient

$$D \approx D_0\exp\left[-\frac{2a}{\hbar}\sqrt{2m(V-E)}\right], \tag{2.146}$$

Therefore $D > 0$. This means that there is some small but still positive probability of finding a particle behind the barrier, in the third region, even when the energy of the particle is less than the energy of the obstacle.

Only one tunnel effect situation is shown here. It may be simpler and therefore more popular, and you can find some other calculations with a classic review of uncertainty relations in my mentioned book [4]. I will only add here that the existence of the tunnel effect shows that the coincidences and uncertainties of the micro world are not the result of our "not knowing all of the conditions or ignorance", as *Laplas* would say about the probability, but of objective coincidences.

2.6.4 Matrix form

We write a matrix representation with common notations in quantum physics. Let the linear operator $\hat{A}$ be determined by the equation

$$\hat{A}|\psi\rangle = |\phi\rangle \tag{2.147}$$

for some vectors $|\phi\rangle$ and $|\psi\rangle$. Then we have:

$$|\phi\rangle = \hat{A}|\psi\rangle = \hat{A}\sum_k |\varphi_k\rangle\langle\varphi_k|\psi\rangle = \sum_k \hat{A}|\varphi_k\rangle\langle\varphi_k|\psi\rangle,$$

$$\langle\varphi_j|\phi\rangle = \sum_k\langle\varphi_j|\hat{A}|\varphi_k\rangle\langle\varphi_k|\psi\rangle,$$

what we write

$$\phi_j = \sum_k a_{jk}\psi_k, \tag{2.148}$$

where $a_{jk} = \langle\varphi_j|\hat{A}|\varphi_k\rangle$ and $\psi_k = \langle\varphi_k|\phi\rangle$ are scalars. Equation (2.147) is equivalent (equal to isomorphism) with

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \dots\\ \psi_n\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \dots\\ \phi_n\end{pmatrix}, \tag{2.149}$$

where $n \to \infty$ can also be. The values of a_{ij} are *matrix elements*, the coefficients of the operator $\hat{A}$ in relation to the base $\{|\varphi_j\rangle; j = 1, 2, \dots, n\}$.

Consider this with the example of a quantum mechanical *oscillator*. *Hamiltonian* of a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2.150}$$

where m is the mass of the particle, k is the constant, $\omega = \sqrt{k/m}$ is the angular frequency of the oscillator, and the momentum and position operators are $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $\hat{x}$ (multiplication by x). The first addition is kinetic energy, the second is potential.

First, let's use the time-independent Schrödinger equation

$$\hat{H}|\psi\rangle = E|\psi\rangle, \tag{2.151}$$

where E is the intrinsic value of energy, and $|\psi\rangle$ is the intrinsic energy state. This differential equation for the inherent problem in the coordinate base, for the wave function $\langle\chi|\psi\rangle = \psi(x)$, can be solved by *spectral method*[32]. The solutions are well known

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\cdot e^{-\frac{m\omega x^2}{2\hbar}}\cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad n = 0, 1, 2, \dots \tag{2.152}$$

where are

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right) \tag{2.153}$$

the Hermite polynomials. The appropriate energy levels are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) = (2n+1)\frac{\hbar}{2}\omega. \tag{2.154}$$

As we can see, the energies of the harmonic oscillator are quantized, which means that only discrete values (integer halves of the number $\hbar\omega$) are possible, which is a general characteristic of quantum-mechanical systems for constrained particles.

Discrete levels are evenly distributed (unlike the Bohr model of the atom), and the lowest energy (zero, ground state) is not equal to the minimum of the potential well but is $\hbar\omega/2$ above. Due to this *zero point energy*, the position and moment of the oscillator in the ground state are not fixed (as it would be in a classical oscillator), but have a small range of deviations, in accordance with Heisenberg's uncertainty principle.

Then we move on to the position and momentum matrices, the functions $\hat{\mathbf{X}}(t)$ and $\hat{\mathbf{P}}(t)$, which are sinusoidal. It is the energy of the oscillator again

$$\hat{\mathbf{H}} = \frac{1}{2}(\hat{\mathbf{P}}^2 + \hat{\mathbf{X}}^2), \tag{2.155}$$

[32]Spectral method: `http://www.scholarpedia.org/article/Spectral_methods`

which is Heisenberg's (matrix) equivalent to Schrödinger's (2.150). Energy levels are in the orbits of the phase space. The classical orbit with energy E is

$$X(t) = \sqrt{2E}\cos t, \quad P(t) = -\sqrt{2E}\sin t. \tag{2.156}$$

Quantization requires that the integral of $P\,dX$ per orbit, the circular surface of the phase space, must be an integer product of the Planck constant. The area of a circle of radius $\sqrt{2E}$ is $2\pi E$, and hence

$$E = \frac{nh}{2\pi}, \tag{2.157}$$

so in *natural units* (defined by universal physical constants), with $\hbar = 1$, energy is an integer.

Fourier's[33] the components of position and momentum are simple, but they are even easier to work with when they are in the expressions:

$$\hat{\mathbf{A}} = \hat{\mathbf{X}}(t) + i\hat{\mathbf{P}}(t) = \sqrt{2E}e^{-it}, \quad \hat{\mathbf{A}}^* = \hat{\mathbf{X}}(t) - i\hat{\mathbf{P}} = \sqrt{2E}e^{it}. \tag{2.158}$$

Both operators, the matrix $\hat{\mathbf{A}}$ and the conjugate $\hat{\mathbf{A}}^\dagger$ have only simple frequencies, and the position matrices ($\hat{\mathbf{X}}$) and momentum ($\hat{\mathbf{P}}$) are obtained by adding and subtracting them.

How $\hat{\mathbf{A}}$ evolves into a classic Fourier series[34] with only the lowest frequency, and its matrix element a_{jk} is the $j-k$ -th Fourier coefficient of the classical orbit, the matrix for $\hat{A}$ is not zero only along the secondary diagonal where it takes the values $\sqrt{2E_n}$. The matrix of the operator $\hat{A}^*$ is also nonzero only on the side diagonal, with the same elements. Reconstruction finds

$$2\hat{\mathbf{X}}(0) = \sqrt{\hbar}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots \end{pmatrix} \tag{2.159}$$

and

$$2\hat{\mathbf{P}}(0) = \sqrt{\hbar}\begin{pmatrix} 0 & -i\sqrt{1} & 0 & 0 & \dots \\ i\sqrt{1} & 0 & -i\sqrt{2} & 0 & \dots \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & \dots \\ \dots \end{pmatrix}. \tag{2.160}$$

These are the *Heisenberg matrices* for a harmonic oscillator. They are Hermitian, because they are obtained from real Fourier coefficients.

Calculating the coefficients of these matrices for some further moment t is simple, because they simply evolve over time

$$x_{jk}(t) = x_{jk}(0)e^{i(E_j-E_k)t}, \quad p_{jk}(t) = p_{jk}(0)e^{i(E_j-E_k)t}. \tag{2.161}$$

The product of position and momentum matrices is not a Hermitian matrix, but has a real and an imaginary part. The real part is half of the *anti-commutator* $\{\hat{\mathbf{X}},\hat{\mathbf{P}}\} = \hat{\mathbf{X}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{X}}$, and the imaginary is proportional to the *commutator* $[\hat{\mathbf{X}},\hat{\mathbf{P}}] = \hat{\mathbf{X}}\hat{\mathbf{P}} - \hat{\mathbf{P}}\hat{\mathbf{X}}$. It is easy to check that this commutator, in the case of a harmonic oscillator, is equal to $i\hbar$, multiplied by a unit matrix (operator). It is also easy to check that the initial matrix (2.155) is diagonal, with eigenvalues E_n, energy levels.

[33] Joseph Fourier (1768-1830), French mathematician and physicist.
[34] Fourier Series: `https://www.math24.net/fourier-series-definition-typical-examples/`

2.6.5 Ladder operators

There are two types of *ladder operators*, creation and annihilation operators. These operators increase or decrease the eigenvalues of other operators. We use them to change the quantum numbers of the angular momentum and to change the energy levels of the quantum harmonic oscillator.

Assume that for the operators $\hat{X}$, $\hat{Y}$ and the commutator $[\hat{X},\hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}$ hold equations:

$$\hat{X}\vec{x} = x\vec{x}, \quad [\hat{X},\hat{Y}] = y\hat{Y}, \tag{2.162}$$

where $\vec{x}$ is the eigenvector of the first, and x and y are scalars. Then:

$$\hat{X}\hat{Y}\vec{x} = (\hat{Y}\hat{X} + [\hat{X},\hat{Y}])\vec{x} = \hat{Y}\hat{X}\vec{x} + [\hat{X},\hat{Y}]\vec{x} = x\hat{Y}\vec{x} + y\hat{Y}\vec{x} = (x+y)\hat{Y}\vec{x},$$

that is

$$\hat{X}\vec{y} = (x+y)\vec{y}. \tag{2.163}$$

The eigenvalue (x) of the first operator ($\hat{X}$) changed by y, where the eigenvector (instead of $\vec{x}$) became $\vec{y} = \hat{Y}\vec{x}$. When y is a positive number then $\hat{Y}$ is lift or *creation operator*, and if y is a negative number then $\hat{Y}$ does descent or *annihilation*.

Example 1. We can construct ladder operators using *Pauli matrices* (2.56), first putting:

$$\hat{S}_x = \frac{1}{2}\hat{\sigma}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{1}{2}\hat{\sigma}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{1}{2}\hat{\sigma}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.164}$$

and then:

$$\hat{S}^+ = \hat{S}_x + i\hat{S}_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}^- = \hat{S}_x - i\hat{S}_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.165}$$

These are the operators *spin*, where $\hat{S}^+$ is the operator of raising and $\hat{S}^-$ of lowering.

The operators $\hat{S}^+$ and $\hat{S}^-$ are not Hermitian, because they change by successive transposition and conjugation. Therefore, they do not directly represent physical values, but they simplify mathematical expressions and are useful for interpreting processes.

It is easy to check that:

$$[\hat{S}_z, \hat{S}^+] = \hat{S}^+, \quad [\hat{S}_z, \hat{S}^-] = -\hat{S}^-, \tag{2.166}$$

and in the way (2.163) for the operator $\hat{S}_z\vec{x} = x\vec{x}$ get $\hat{S}_z\hat{S}^+\vec{x} = (x+1)\hat{S}^+\vec{x}$, i.e. $\hat{S}_z\vec{y} = (x+1)\vec{y}$, where $\vec{y} = \hat{S}^+\vec{x}$. In the same way $\hat{S}_z\hat{S}^-\vec{x} = (x-1)\hat{S}^-\vec{x}$, or $\hat{S}\vec{y}' = (x-1)\vec{y}'$, with $\vec{y}' = \hat{S}^-\vec{x}$.

There are only slightly more complex things in quantum physics, where for a fixed spin or intensity of the orbital momentum S the lifting operator $\hat{S}^+$ moves the eigenstate $\psi_{s,m}$ up to the next eigenstate $\psi_{s,m+1}$, and the descent operator $\hat{S}^-$ moves the eigenstate $\psi_{s,m}$ down to $\psi_{s,m-1}$.

Example 2. Recall a harmonic oscillator who's Hamiltonian is (2.150). We define linear combinations of the position operator $\hat{x}$ and the momentum $\hat{p} = -id/dx$ as two new operators:

$$\hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}), \quad \hat{a}^- = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \tag{2.167}$$

which can also be shown to be lifting and lowering operators. Unlike the position and momentum operators themselves, the operators $\hat{a}^+$ and $\hat{a}^-$ are not Hermitian and as such do not represent observable (physically measurable quantities).

Their product is almost Hamiltonian:

$$2\hat{a}^+\hat{a}^- = \hat{x}^2 + \hat{p}^2 - i(\hat{p}\hat{x} - \hat{x}\hat{p}) = 2\hat{H} - i[\hat{p}, \hat{x}] = 2\hat{H} - 1,$$

$$2\hat{a}^-\hat{a}^+ = \hat{x}^2 + \hat{p}^2 + i(\hat{p}\hat{x} - \hat{x}\hat{p}) = 2\hat{H} + i[\hat{p}, \hat{x}] = 2\hat{H} + 1,$$

because the commutator is $[\hat{p}, \hat{x}] = -i\hbar$, that is:

$$\hat{a}^+\hat{a}^- = \hat{H} - \frac{1}{2}, \quad \hat{a}^-\hat{a}^+ = \hat{H} + \frac{1}{2}, \tag{2.168}$$

in a system of units with $\hbar = 1$. Hence, from $\hat{H}\psi = E\psi$ we get:

$$\hat{H}(\hat{a}^+\psi) = (\hat{a}^+\hat{a}^- + \frac{1}{2})(\hat{a}^+\psi) = \hat{a}^+(\hat{a}^-\hat{a}^+ + \frac{1}{2})\psi =$$

$$= \hat{a}^+(\hat{H} + 1)\psi = \hat{a}^+(E + 1)\psi = (E + 1)(\hat{a}^+\psi),$$

or:

$$\hat{H}(\hat{a}^+\psi) = (E + 1)(\hat{a}^+\psi), \quad \hat{H}(\hat{a}^-\psi) = (E - 1)(\hat{a}^-\psi), \tag{2.169}$$

where the second equation is obtained in a similar way to the first. Therefore, $\hat{a}^+\psi$ is an eigenfunction of energy that is exactly one step larger than ψ itself. In common units, this energy step is $\hbar\omega$. There is no limit in this kind of climbing for a harmonic oscillator, but when descending you cannot go lower than the basic energy state $\frac{1}{2}\hbar\omega$.

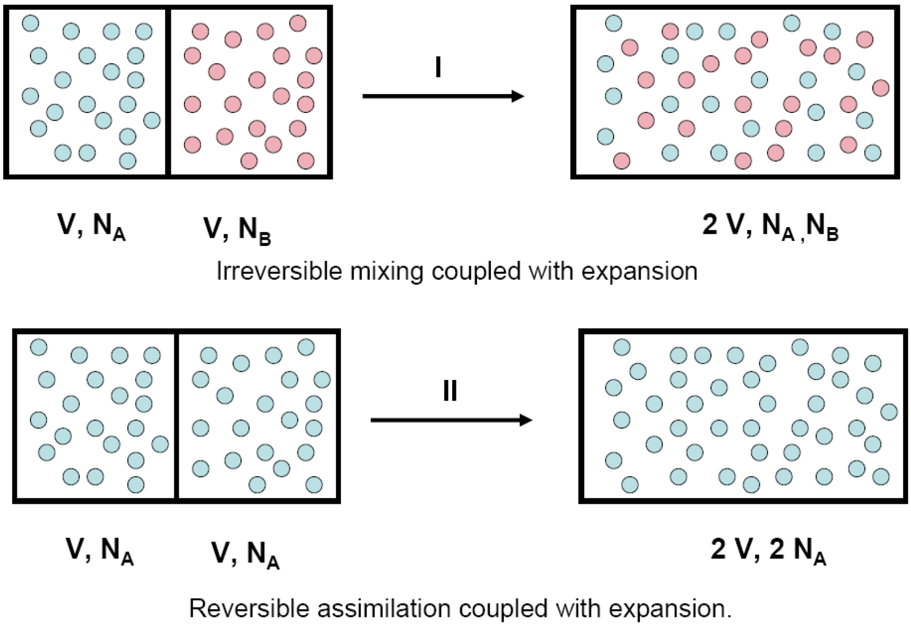
As we know, in (this) orthonormal base, the matrix representations of the lifting and lowering operators for a quantum harmonic oscillator are:

$$\hat{a}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots \\ \dots \end{pmatrix}, \quad \hat{a}^- = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots \\ \dots \end{pmatrix}, \tag{2.170}$$

which follows from $a^+_{jk} = \langle\psi_j|\hat{a}^+|\psi_k\rangle$ and $a^-_{jk} = \langle\psi_j|\hat{a}^-|\psi_k\rangle$, analogous to (2.149).

2.7 Second quantization

Second quantization is a formalism used to describe and analyze quantum systems of many bodies. In quantum field theory, it is also called *canonical quantization*, in which fields (usually as wave functions of matter) are considered field operators with properties similar to physical quantities (position, momentum, etc.), otherwise operators of the first quantization. The key ideas of this method were introduced by Paul *Dirac* (1927), and developed primarily by Vladimir *Fock* and Pascual *Jordan*.



Slika 2.10: Gibbs' paradox.

2.7.1 Gibbs' paradox II

The paradox of *Gibbs*[35] from 1875 occurs in the *entropy* of statistical mechanics when all gas particles are considered different in a situation when they are no longer. The solution to the paradox is to treat particles of the same gas indistinguishable, such that the state of the system does not change during the *permutation* of two particles. In situations like the one in the figure 2.10, we first consider this briefly. index entropy index permutation

In the upper part of the picture are two parts of an insulated vessel of the same volume V filled with an ideal gas of the same temperature T, with and without a baffle. Let one mole of gas A be in the left part, and one mole of gas in B the right part, or as in the picture in order with the number of positions N_A or N_B, entropy S_A and S_B. When we remove the barrier, the gases mix and spread over the entire volume of $2V$, so the available volume for individual gases has doubled. The total entropy is increased by $\Delta S = 2R\ln 2$, where R is the universal gas constant. It is an example of an irreversible process.

In the bottom row of that picture, there is the same gas in both parts of the vessel. By removing the barrier, the "two gases" do not mix and do not expand in volumes twice their size, and it is possible to simply return the barrier to the state of the right to the state of the left. This is an example of a return process, and if the same argumentation were valid as in the previous case, we would have a contradiction, because now there is no change in entropy ($\Delta S = 0$).

Examples indicate that entropy (in microcanonical ensemble, as well as in case of absence of interactions) is not an *extensive quantity* (proportional to the amount of a substance), but depends on the possible arrangement of particles so that it is independent of their order. It is a function of combinations rather than variations of layouts. Therefore, the factor $1/N!$ Should be added to the entropy, where N is the number of particle divisions and equal particles are considered indistinguishable. Aside from the question of where that number

[35] Josiah Willard Gibbs (1839-1903), American scientist.

comes from, let's see that the account of the situation from the given picture is now correct.

I note that there are opinions that this "paradox" is given too much importance, as in the appendix (see [11]) from which the mentioned picture was taken, but the discussion about it is again interesting due to the implications in interpreting the probability and statistics, and then alleged (my) information theories.

Consider an ideal gas of N particles in a vessel of volume V. The vessel is divided into two sections of volume V_1 and V_2, so that $V_1 + V_2 = V$. In the first section the number of particles is N_1 and in the second N_2 so that $N_1 + N_2 = N$, and it is assumed that the density of particles in both sections is the same $\rho = N_1/V_1 = N_2/V_2$.

When the barrier between the sections is removed and the particles are identical, the total entropy should not increase. However, if the entropies in the compartments before removing the barrier were in order:

$$S_1 \sim N_1 k \ln V_1 + \frac{3}{4} N_1 k, \quad S_2 \sim N_2 k \ln V_2 + \frac{3}{4} N_2 k, \tag{2.171}$$

with the cumulative value $S = S_1 + S_2$. After removing the barrier the total entropy is

$$S' \sim (N_1 + N_2) k \ln(V_1 + V_2) + \frac{3}{2}(N_1 + N_2) k. \tag{2.172}$$

The difference between the total entropies after and before the removal of the barrier is

$$\Delta S = S' - S = (N_1 + N_2) k \ln(V_1 + V_2) - N_1 k \ln V_1 - N_2 k \ln V_2,$$

therefore

$$\Delta S = N_1 k \ln \frac{V}{V_1} + N_2 k \ln \frac{V}{V_2} > 0. \tag{2.173}$$

Conclusion $\Delta S > 0$ contradicts the assumption that the total entropy of a closed system (sufficiently wider system of these vessels) is constant, so it is also contrary to the law of information conservation (increase of entropy is loss of information).

Let us now introduce $\ln N!$. Stirling's approximation gives $\ln N! \approx N \ln N - N$, so for entropy we have

$$S = Nk \ln \left[\frac{V}{Nh^3} \left(\frac{2\pi m}{\beta} \right)^{3/2} \right] + \frac{5}{2} Nk, \tag{2.174}$$

version of the well-known Sackur-Tetrode equation. I remind you, k is the Boltzmann constant, x is the Planck constant, m is the mass of the particle, and details such as the thermal wavelength Λ and the internal energy of the gas U are omitted, which do not affect the point. Now the entropy increment becomes:

$$\begin{aligned} \Delta S = (N_1 + N_2) k \ln \frac{V_1 + V_2}{N_1 + N_2} - N_1 k \ln \frac{V_1}{N_1} - N_2 k \ln \frac{V_2}{N_2} = \\ = N_1 k \ln \frac{V}{V_1} + N_2 k \ln \frac{V}{V_2} - N_1 k \ln \frac{N}{N_1} - N_2 k \ln \frac{N}{N_2} \\ = N_1 k \ln \frac{V N_1}{N V_1} + N_2 k \ln \frac{V N_2}{N V_2} + 0, \end{aligned}$$

because due to the constant density $\rho = N_1/V_1 = N_2/V_2 = N/V$ the numerics of the logarithms are ones, the logarithms are zero, so the change in entropy is also zero ($\Delta S = 0$). Therefore, with the help of the $1/N!$ member, we avoided the Gibbs paradox. And that is already a quantum mechanical treatment of equal molecules of an ideal gas.

2.7.2 Indistinguishability

One path from Gibbs' paradox leads to the probability distribution theorems I dealt with in the book "Physical Information" (see [3]), and the other to the *elementary particles* of physics in the way that is the topic here.

Identical particles in quantum mechanics are those that do not differ in mass, charge, spin, isospin, or any other intrinsic characteristic. These are the postulate *indistinguishability*, which says that identical particles cannot be distinguished by any measurement, which is a theoretical extreme that stands in quantum mechanics due to uncertainty relations.

When one particle is in the state n_1 and the other in the state n_2, we have the quantum state of the system which we write with

$$|n_1\rangle|n_2\rangle, \tag{2.175}$$

where the order of writing is also important, so $|n_2\rangle|n_1\rangle$ means that the first particle takes the state $|n_2\rangle$ and the second the state $|n_1\rangle$. It is a canonical way of writing the database of the tensor product of the space $X \otimes X$, a combined system of individual spaces X.

The two states, $|n_1\rangle|n_2\rangle$ and $|n_2\rangle|n_1\rangle$, are physically equivalent only if they differ in a complex phase factor. For two indistinguishable particles, the state before the exchange of particles must be physically equivalent to the state after exchange, so that these two states differ only in (complex) phase factors. This fact suggests that the state for two indistinguishable (and non-interacting) particles is written in the following two ways:

$$|n_1\rangle|n_2\rangle \pm |n_2\rangle|n_1\rangle. \tag{2.176}$$

The state after the "plus" sign is symmetrical, and with the "minus" sign it is called antisymmetric. More fully written these states are:

$$\begin{cases} |n_1, n_2, S\rangle = \lambda_S(|n_1\rangle|n_2\rangle + |n_2\rangle|n_1\rangle), & \text{symmetrically,} \\ |n_1, n_2, A\rangle = \lambda_A(|n_1\rangle|n_2\rangle - |n_2\rangle|n_1\rangle), & \text{antisymmetrically,} \end{cases} \tag{2.177}$$

where the scalars λ_S and λ_A are some constants. Note that the antisymmetric state will be zero if n_1 and n_2 are the same state, and this cannot give a quantum state because it cannot be normalized. In other words, more than one identical particle cannot be in some antisymmetric state. This is the familiar *Pauli exclusion principle*.

It is not possible to distinguish, say, two electrons, they are identical in every way. So there are such particles. Denote the *substitution operator* of particle locations by $\hat{P}_{12}$, so that $\hat{P}_{12}\psi(x_1, x_2) = \psi(x_2, x_1)$, so $\hat{P}_{12}\hat{P}_{12}\psi(x_1, x_2) = \psi(x_1, x_2)$. Therefore, this operator applied twice returns the quantum state to the initial value, which means that the eigenvalues of this operator are only two possible, +1 and −1, so

$$\hat{P}_{12}\psi_\pm = \pm\psi_\pm. \tag{2.178}$$

It turns out that both of these values exist in nature. Particles with the quantum number −1 are like electrons, they are called *fermions* and their spin is a half-integer. Particles with this number +1 are like electrons, they are called *bosons* and their spin is an integer.

Translated into the language of isometric transformations, which are always some rotations, the substitution operator can also be defined as follows

$$\psi(x_1, x_2) = e^{i\alpha}\psi(x_2, x_1), \tag{2.179}$$

where α is a real constant. Repeating the transformation, we return to the initial state, and multiply the function ψ by $e^{2\alpha}$. So $e^{2i\alpha} = 1$, or $e^{i\alpha} = \pm 1$, thence

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1), \tag{2.180}$$

which comes back to (2.178).

Hence the more general conclusions. *Wave function* can have only two possibilities: either it is symmetric (it does not change by changing the place of particles), or it is antisymmetric (by changing the place of particles it changes sign). Additionally, all wave functions of a given system must have the same symmetry; otherwise their superposition could be different from both symmetry and anatisymmetry. Therefore, particle systems can only be either symmetric or antisymmetric, so there are only two types of Fermi-Dirac *statistics* valid for fermions and Bose-Einstein for bosons.

The Hamiltonian, the energy operator, is symmetric

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \hat{U}(|x_1 - x_2|) + \hat{V}(x_1) + \hat{V}(x_2), \tag{2.181}$$

which means that the energy behaves like a boson. On the other hand, the state of fermions, which in quantum physics is written like this

$$\psi = u_k(x_1)u_j(x_2) \to u_i(x_1)u_j(x_2) - u_j(x_1)u_k(x_2), \tag{2.182}$$

becomes zero ($\psi = 0$) when (any) two fermions are in the same state ($x_1 = x_2$) and zero cannot be normalized, which means that such a wave function cannot be a superposition and cannot have physical meaning. This is a conclusion equal to that after (2.177), that the Pauli principle of exclusion applies to fermions. Any two fermions cannot be in the same state.

In addition to the interpretation of energy (2.181), that it is a boson type (like a photon), then the property of elementary particles to be either bosons or fermions, we conclude that fermions are substantial (material), such as electrons and protons. In addition, fermions can be in conjunction with anti-fermions and must be created in pairs, while bosons are in that sense alone, they are their own anti-particles as is the case with light.

Dirac was the first to use lifting and lowering operators in his theory of the electromagnetic field (1927), and in the same year Jordan and *Klein*[36], and in the following *Wigner*[37] used Dirac's description for a multiparticle system in which particles can interact.

2.7.3 Fock spaces

Let $\mathcal{H}$ be a complex Hilbert space and $n \in \mathbb{N}$ an arbitrary natural number. Consider an n-fold *tensor product* of space

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}, \tag{2.183}$$

and for the n-tour of the vector $u_1, \dots, u_n \in \mathcal{H}$ we define: a symmetric tensor product

$$u_1 \vee \cdots \vee u_n = \frac{1}{n!} \sum_\pi u_{\pi(1)} \otimes \cdots \otimes u_{\pi(n)}, \tag{2.184}$$

[36]Oskar Klein (1894-1977), a Swedish theoretical physicist.

[37]Eugene Wigner (1902-1995), a Hungarian-American theoretical physicist and mathematician.

and an antisymmetric tensor product

$$u_1 \wedge \dots \wedge u_n = \frac{1}{n!} \sum_{\pi} \varepsilon_\pi u_{\pi(1)} \otimes \dots \otimes u_{\pi(n)}, \tag{2.185}$$

where it is added to all $n! = 1 \cdot 2 \cdot 3 \cdots n$ permutation of the index string, and ε_π is the sign (± 1) of the permutation π. Thus we obtain Fock[38] spaces using Hilbert spaces.

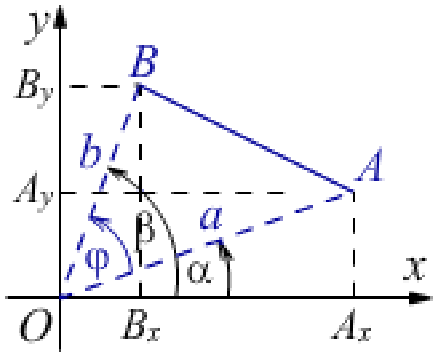
The closed subspace $\mathcal{H}^{\otimes n}$ generated by the product $u_1 \vee \dots \vee u_n$ is denoted by $\mathcal{H}^{\vee n}$, and generated by $u_1 \wedge \dots \wedge u_n$ denote $\mathcal{H}^{\wedge n}$. The first is called the n-fold symmetric tensor product of Hilbert spaces, and the second the n-fold antisymmetric tensor product $(\mathcal{H})$.

Free (or full) *Fock space* over $\mathcal{H}$ is denoted by $\Gamma_\otimes(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\oplus n}$, symmetric (or bosonic) Fock space is $\Gamma_\vee(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}$, and antisymmetric (fermionic) $\Gamma_\wedge(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n}$. Each of them is supplied with its own scalar product.

The simplest case of a symmetric Fock space is obtained by putting $\mathcal{H} = \mathbb{C}$ and then $\Gamma_\vee(\mathbb{C}) = \ell^2(\mathbb{N})$. Symmetrical Fock space is never finally dimensional. When $\mathcal{H}$ is of finite dimension n, then $\mathcal{H}^{\wedge m} = 0$ for $m > n$, so $\Gamma_\wedge(\mathcal{H})$ has dimensions 2^n. In physics, bosonic and fermionic spaces over $\mathcal{H} = L^2(\mathbb{R}^3)$ are most often considered.

Examples of commutators

In the figure 2.11 in Cartesian rectangular system Oxy the points $A(A_x, A_y)$ and $B(c_x, c_y)$ are given which represent the vertices of the vector $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$, intensity $a = |\vec{a}|$ and $b = |\vec{b}|$, deviation from abscissa $\alpha = \angle(xOA)$ and $\beta = \angle(xOB)$, with an angle between them $\varphi = \beta - \alpha$.



Slika 2.11: Vector product.

From the picture, for the *commutator* coordinate $[A, B] = A_x B_y - B_x A_y$ we can easily find:

$$[A, B] = (a \cos\alpha)(b \sin\beta) - (b \cos\beta)(a \sin\alpha) = ab \sin(\beta - \alpha),$$

from where

$$[A, B] = ab \sin\varphi = \vec{a} \wedge \vec{b}, \tag{2.186}$$

which is also called *pseudo scalar product*. Similarly we find

$$\{A, B\} = A_x B_y + B_x A_y = ab \sin(\beta + \alpha), \tag{2.187}$$

which is called *anti-commutator*. The commutator corresponds to the determinant of Fock space, and the anti-commutator to *permanent* matrix (determinants without minuses).

[38] see Fock, Z. Phys. 75 (1932), 622-647

Position and momentum

In classical mechanics, the system of $n = 1, 2, 3, \dots$ material points describe their positions $Q_k(t)$ and momentums $P_k(t)$ for $k \in \{1, 2, \dots, n\}$ at time t. The function $H(P, Q)$ so-called *Hamiltonian* represents the total energy of the system and satisfies the equations:

$$\frac{\partial H}{\partial P_k} = \dot{Q}_k, \quad \frac{\partial H}{\partial Q_k} = -\dot{P}_k, \quad k = 1, 2, \dots n. \tag{2.188}$$

If $f(P, Q)$ is a function of the trajectory, then it is

$$\frac{df}{dt} = \sum_{k=1}^{n} \frac{\partial f}{\partial P_k}\frac{\partial P_k}{\partial t} + \frac{\partial f}{\partial Q_k}\frac{\partial Q_k}{\partial t}, \tag{2.189}$$

that is

$$\frac{df}{dt} = \{H, f\}, \tag{2.190}$$

where the anti-commutator (2.187), now a function, is reformulated into

$$\{h, g\} = \sum_k \frac{\partial h}{\partial P_k}\frac{\partial g}{\partial Q_k} - \frac{\partial h}{\partial Q_k}\frac{\partial g}{\partial P_k}, \tag{2.191}$$

which is analogous to the commutator (2.186). In particular (for indices $k, j = 1, 2, \dots, n$) is:

$$\{P_k, P_j\} = \{Q_k, Q_j\} = 0, \quad \{P_k, Q_j\} = \delta_{kj}, \tag{2.192}$$

where $\delta_{kk} = 1$ for every k, and $\delta_{kj} = 0$ for $k \ne j$, which is easy to check.

In quantum mechanics, the situation is essentially the same. We have a self- adjoint operator $\hat{H}$ (Hamiltonian) that describes all state evolutions via the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = \hat{H}\psi(t). \tag{2.193}$$

Also, we have self-adjoint operators $\hat{Q}_k$ and $\hat{P}_k$ which represent positions and momentums and which evolve in the following way:

$$\hat{Q}_k(t) = e^{i\hbar t\hat{H}}\hat{Q}_k e^{-i\hbar t\hat{H}}, \quad \hat{P}_k(t) = e^{i\hbar t\hat{H}}\hat{P}_k e^{-i\hbar t\hat{H}}. \tag{2.194}$$

Any observables A of the system satisfies the equation (commutator) of evolution

$$\frac{d}{dt}A(t) = -\frac{i}{\hbar}[A(t), \hat{H}], \tag{2.195}$$

so again we have (appropriate) relations now for the commutators:

$$[\hat{P}(x), \hat{P}(x)] = [\hat{Q}(x), \hat{Q}(x)] = 0, \quad [\hat{Q}(x), \hat{P}(x)] = i\hbar\delta(x - y)\hat{I}. \tag{2.196}$$

These define the Schrödinger equation (2.193), and (2.192) define the Hamiltonian (2.188), i.e. the total energy of the classical system.

Analogous (2.158) if we define operators:

$$\hat{a}(x) = \frac{1}{\sqrt{2}}(\hat{Q}(x) + i\hat{P}(x)), \quad \hat{a}^*(x) = \frac{1}{\sqrt{2}}(\hat{Q}(x) - i\hat{P}(x)), \tag{2.197}$$

then $\hat{a}(x)$ and $\hat{a}^*(x)$ are adjoint, and:

$$[\hat{a}(x), \hat{a}(y)] = [\hat{a}^*(x), \hat{a}^*(y)] = 0, \quad [\hat{a}(x), \hat{a}^*(y)] = \hbar\delta(x-y)\hat{I}, \tag{2.198}$$

and these are the *canonical relations* of the commutators. Formulations of the uniqueness of canonical relations between position and momentum operators are called Stone-Neumann theorem[39], the name given to it after Harvey *Stone*[40] and John von *Neumann*[41].

Determinant

The scalar product of antisymmetric tensor products becomes the sum of all permutations (π and σ) of the index series $1, 2, \dots, n$

$$\langle u_1 \wedge \dots \wedge u_n | v_1 \wedge \dots \wedge v_n \rangle = \frac{1}{(n!)^2} \sum_{\pi,\sigma} \varepsilon_\pi \varepsilon_\sigma \langle u_{\pi(1)} | v_{\sigma(1)} \rangle \dots \langle u_{\pi(n)} | v_{\sigma(n)} \rangle, \tag{2.199}$$

where ε_π and ε_σ are the signs ($\pm$) of the permutation (even and odd). This can be represented in the form of a determinant

$$\langle u_1 \wedge \dots \wedge u_n | v_1 \wedge \dots \wedge v_n \rangle = \frac{1}{n!} \det(w_{jk}), \tag{2.200}$$

matrices whose coefficients are $w_{jk} = \langle u_j | v_k \rangle$ for $j, k = 1, 2, \dots, n$.

When we replace the scalar product of the space $\mathcal{H}^{\otimes n}$ with the scalar product of the subspace $\mathcal{H}^{\wedge n}$, the factorial $n!$ Disappears in the previous determinant, so we write

$$\langle u_1 \wedge \dots \wedge u_n | v_1 \wedge \dots \wedge v_n \rangle_\wedge = \det(w_{jk}), \tag{2.201}$$

where is

$$||u_1 \wedge \dots \wedge u_n||_\wedge^2 = n! ||u_1 \wedge \dots \wedge u_n||_\otimes^2. \tag{2.202}$$

On the left side of the equation is the norm of the subspace $\mathcal{H}^{\wedge n}$, on the right side is the norm of the space $\mathcal{H}^{\otimes n}$.

In the same way in the subspace of symmetric tensor products $\mathcal{H}^{\vee n}$ we get

$$\langle u_1 \vee \dots \vee u_n | v_1 \vee \dots \vee v_n \rangle_\vee = \mathrm{per}\,(w_{jk}), \tag{2.203}$$

where $\mathrm{per}\,(w_{jk}) = \det_\vee(w_{jk})$ denotes the emph permanent matrix, the determinant without minus in front of the cofactor. By analogy,

$$||u_1 \vee \dots \vee u_n||_\vee^2 = n! ||u_1 \vee \dots \vee u_n||_\otimes^2, \tag{2.204}$$

for the norms of the subspace $\mathcal{H}^{\vee n}$ and the space $\mathcal{H}^{\otimes n}$.

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[39] Stone – Neumann theorem: `https://arxiv.org/pdf/0912.0574.pdf`

[40] Marshall Harvey Stone (1903-1989), American mathematician.

[41] John von Neumann (1903-1957), Hungarian-American mathematician and physicist.

Symmetrical spaces

Here we mean a symmetric Fock space $\Gamma_\vee(\mathcal{H})$. Note that in such a space the equation $u\vee\dots\vee u = u\otimes\dots\otimes u$ holds. The exponential, or *coherent vector* (see [10]) associated with u is

$$\kappa(u) = \sum_{n\in\mathbb{N}} \frac{1}{n!} u^{\otimes n}, \tag{2.205}$$

so for a scalar product in $\Gamma_\vee$ we get

$$\langle \kappa(u)|\kappa(v)\rangle = \exp\langle u|v\rangle. \tag{2.206}$$

Denote by $\mathcal{K}$ the space of finitely linear combinations of coherent vectors.

Scalar product $\langle u|v\rangle$ of two vectors $(u, v\in\mathcal{H})$ is a number from zero to one which we can say represents their *fidelity*. In case one of them has the direction of the coordinate axis, their product gives the probability of the expression of the other on that axis, the measurement is observable. The probability is reciprocal to some mean value of the number of options and the logarithm of that number is the information. Therefore, the scalar product of coherent vectors on the left side of the equation (2.206) refers to the scalar product in the exponent as "number of options" to "information".

Lemma 2.7.1. *Each finite family of coherent vectors is linearly independent.*

Proof. Let $u_1,\dots,u_n\in\mathcal{H}$ be given. Then $U_{jk} = \{u\in\mathcal{H} : \langle u|u_j\rangle \neq \langle u|u_k\rangle\}$ for $j\neq k$ dense open sets in $\mathcal{H}$. Their intersection $\bigcap_{jk} U_{jk}$ is not an empty set, which means that there exists $v\in\mathcal{H}$ so that the numbers $\alpha_k = \langle v|u_k\rangle$ are separated. The existence of scalars β_j such that $\sum_{j=1}^n \beta_j\kappa(u_j) = 0$ would lead to

$$0 = \langle\kappa(zv)|\sum_{j=1}^n \beta_j\kappa(u_j)\rangle = \sum_{j=1}^n \beta_j e^{z\alpha_j},$$

for all $z\in\mathbb{C}$. However, we know that the functions $z\to\exp(\alpha_j z)$ are linearly independent, so all β_j must be equal to zero. This proves that the family $\{\kappa(u_1),\dots,\kappa(u_n)\}$ consists of linearly independent (coherent) vectors. □

Lemma 2.7.2. *Space $\mathcal{K}$ is dense in $\Gamma_\vee(\mathcal{H})$.*

Proof. Equality

$$u_1\vee\dots\vee u_n = \frac{1}{2^n}\sum_{\varepsilon_j=\pm1} \varepsilon_1\dots\varepsilon_n(\varepsilon_1 u_1+\dots+\varepsilon_n u_n)^{\vee n}$$

show that the set $\{u^{\vee n} : u\in\mathcal{H},\ n\in\mathbb{N}\}$ is complete in $\Gamma_\vee(\mathcal{H})$. The consequence is that equality

$$u^{\vee n} = \frac{d^n}{dt^n}\kappa(tu)\Big|_{t=0}$$

shows that the space $\mathcal{K}$ is dense. □

These two lemmas open up the possibility of modeling real physical space using symmetric Fock. The first then speaks of the separation of the cells of space, and the second that this space is *everywhere dense* in the sense of the metric spaces mentioned earlier.

Corollary 2.7.3. *If $S\subset\mathcal{H}$ is a dense subspace, then the space $\mathcal{K}(S)$ is generated with $\kappa(u)$, $u\in S$, dense in $\Gamma_\vee(\mathcal{H})$.*

Proof. From
$$\|\kappa(u)-\kappa(v)\|^2 = e^{\|u\|^2} + e^{\|v\|^2} - 2\Re\left(e^{(u,v)}\right)$$
follows the continuity of the mapping $u \to \kappa(u)$. Then call the lemmas. □

There are also examples of subsets $S \subset \mathcal{H}$ that are not dense in $\mathcal{H}$ and yet are dense in $\Gamma_\vee(\mathcal{H})$. One such (non-trivial) example in the case of $\mathcal{H} = L^2(\mathbb{R})$ is the set of S indicators of the functions of the Borel set.

2.8 Matrix factorization

The matrices $\hat{A}$ and $\hat{C}$ are given, and we look for *matrix* $\hat{X}$ such that $\hat{A}\hat{X} = \hat{C}$, or:
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}. \tag{2.207}$$
By matrix multiplication we get a system of linear equations:
$$\begin{cases} a_1x_1 + a_2x_3 = c_1, & \hat{C}(1,1), \\ a_1x_2 + a_2x_4 = c_2, & \hat{C}(1,2), \\ a_3x_1 + a_4x_3 = c_3, & \hat{C}(2,1), \\ a_3x_2 + a_4x_4 = c_4, & \hat{C}(2,2). \end{cases} \tag{2.208}$$
The determinant of this system is:
$$D = \begin{vmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & a_3 & 0 & a_4 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & 0 & a_2 \\ 0 & a_4 & 0 \\ a_3 & 0 & a_4 \end{vmatrix} + a_2 \begin{vmatrix} 0 & a_1 & a_2 \\ a_3 & 0 & 0 \\ 0 & a_3 & a_4 \end{vmatrix} = a_1a_4 \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - a_2a_3 \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix},$$
that is
$$D = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}^2 = (\det \hat{A})^2. \tag{2.209}$$
For a unitary matrix, $\det \hat{A} \ne 0$, so $D \ne 0$ and the system (2.208) always has a solution.

The determinant of the first variable (x_1) is:
$$D_1 = \begin{vmatrix} c_1 & 0 & a_2 & 0 \\ c_2 & a_1 & 0 & a_2 \\ c_3 & 0 & a_4 & 0 \\ c_4 & a_3 & 0 & a_4 \end{vmatrix} =$$
$$= c_1 \begin{vmatrix} a_1 & 0 & a_2 \\ 0 & a_4 & 0 \\ a_3 & 0 & a_4 \end{vmatrix} - c_2 \begin{vmatrix} 0 & a_2 & 0 \\ 0 & a_4 & 0 \\ a_3 & 0 & a_4 \end{vmatrix} + c_3 \begin{vmatrix} 0 & a_2 & 0 \\ a_1 & 0 & a_2 \\ a_3 & 0 & a_4 \end{vmatrix} - c_4 \begin{vmatrix} 0 & a_2 & 0 \\ a_1 & 0 & a_2 \\ 0 & a_4 & 0 \end{vmatrix}$$
$$= c_1a_4 \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_2 \cdot 0 + c_3(-a_2) \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_4 \cdot 0,$$
apropos
$$D_1 = (c_1a_4 - c_3a_2) \cdot \det \hat{A}, \tag{2.210}$$

and from there

$$x_1 = (c_1a_4 - c_3a_2)/\det\hat{A}, \tag{2.211}$$

due to the *Cramer*[42] rule $x_1 = D_1/D$.

The determinant of the second variable (x_2) is:

$$D_2 = \begin{vmatrix} a_1 & c_1 & a_2 & 0 \\ 0 & c_2 & 0 & a_2 \\ a_3 & c_3 & a_4 & 0 \\ 0 & c_4 & 0 & a_4 \end{vmatrix} =$$

$$= -c_1\begin{vmatrix} 0 & 0 & a_2 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_4 \end{vmatrix} + c_2\begin{vmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_4 \end{vmatrix} - c_3\begin{vmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & a_4 \end{vmatrix} + c_4\begin{vmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_2 \\ a_3 & a_4 & 0 \end{vmatrix}$$

$$= -c_1\cdot 0 + c_2a_4\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_3\cdot 0 + c_4(-a_2)\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix},$$

regarding

$$D_2 = (c_2a_4 - c_4a_2)\cdot\det\hat{A}, \tag{2.212}$$

and from there

$$x_2 = (c_2a_4 - c_4a_2)/\det\hat{A}, \tag{2.213}$$

because $x_2 = D_2/D$.

The determinant of the third variable (x_3) is:

$$D_3 = \begin{vmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_1 & c_2 & a_2 \\ a_3 & 0 & c_3 & 0 \\ 0 & a_3 & c_4 & a_4 \end{vmatrix} =$$

$$= c_1\begin{vmatrix} 0 & a_1 & a_2 \\ a_3 & 0 & 0 \\ 0 & a_3 & a_4 \end{vmatrix} - c_2\begin{vmatrix} a_1 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & a_3 & a_4 \end{vmatrix} + c_3\begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{vmatrix} - c_4\begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_1 & a_2 \\ a_3 & 0 & 0 \end{vmatrix}$$

$$= c_1(-a_3)\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_2\cdot 0 + c_3a_1\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_4\cdot 0,$$

respecting

$$D_3 = (c_3a_1 - c_1a_3)\cdot\det\hat{A}, \tag{2.214}$$

and from there

$$x_3 = (c_3a_1 - c_1a_3)/\det\hat{A}, \tag{2.215}$$

because $x_3 = D_3/D$.

The determinant of the fourth variable (x_4) is:

$$D_4 = \begin{vmatrix} a_1 & 0 & a_2 & c_1 \\ 0 & a_1 & 0 & c_2 \\ a_3 & 0 & a_4 & c_3 \\ 0 & a_3 & 0 & c_4 \end{vmatrix} =$$

[42] Gabriel Cramer (1704-1752), Swiss mathematician.

$$= -c_1 \begin{vmatrix} 0 & a_1 & 0 \\ a_3 & 0 & a_4 \\ 0 & a_3 & 0 \end{vmatrix} + c_2 \begin{vmatrix} a_1 & 0 & a_2 \\ a_3 & 0 & a_4 \\ 0 & a_3 & 0 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & 0 & a_2 \\ 0 & a_1 & 0 \\ 0 & a_3 & 0 \end{vmatrix} + c_4 \begin{vmatrix} a_1 & 0 & a_2 \\ 0 & a_1 & 0 \\ a_3 & 0 & a_4 \end{vmatrix}$$

$$= -c_1 \cdot 0 + c_2(-a_3) \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} - c_3 \cdot 0 + c_4 a_1 \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix},$$

respectively

$$D_4 = (c_4 a_1 - c_2 a_3) \cdot \det \hat{A}, \tag{2.216}$$

and from there

$$x_4 = (c_4 a_1 - c_2 a_3) / \det \hat{A}, \tag{2.217}$$

because $x_4 = D_4/D$.

We get all this directly from the matrix equation $\hat{A}\hat{X} = \hat{C}$ for the regular matrix $\hat{A}$ by multiplying it on the left with inverse matrix $\hat{A}^{-1} \neq 0$. Hence, and on the basis of (2.207) it follows

$$\hat{X} = \hat{A}^{-1}\hat{C}, \quad \det \hat{A} \neq 0, \tag{2.218}$$

and then:

$$\hat{X} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \frac{1}{\det \hat{A}} \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix},$$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \frac{1}{\det \hat{A}} \begin{pmatrix} a_4 c_1 - a_2 c_3 & a_4 c_2 - a_2 c_4 \\ -a_3 c_1 + a_1 c_3 & -a_3 c_2 + a_1 c_4 \end{pmatrix},$$

or:

$$\begin{cases} x_1 = (c_1 a_4 - c_3 a_2)/\det \hat{A}, & \hat{X}(1,1), \\ x_2 = (c_2 a_4 - c_4 a_2)/\det \hat{A}, & \hat{X}(1,2), \\ x_3 = (c_3 a_1 - c_1 a_3)/\det \hat{A}, & \hat{X}(2,1), \\ x_4 = (c_4 a_1 - c_2 a_3)/\det \hat{A}, & \hat{X}(2,2). \end{cases} \tag{2.219}$$

and this is exactly the same as the previous solutions (2.211), (2.213), (2.215) and (2.217).

Since for a given matrix $\hat{C}$ there are innumerable ways to specify the matrix $\hat{A}$ such that $\det \hat{A} = 1$ and according to the exposed factoring of the matrix $\hat{C}$, we conclude that arbitrary quantum evolution can be represented in countless ways by means of compositions, steps, or sub-evolutions of a given process.

Epilogue

No matter how simple it seemed to me at the beginning that I would retell "several" consequences of the principles of (my) information theory, with each sequel the goal moved away. To link theory only to mathematics or physics, and to omit, for example, biology, psychology, sociology, or law, seemed insufficient, as, say, to limit the use of computers to solving mathematical problems. The specificity of the settings, their novelty and the breadth of the topics they refer to did not allow me to enter into possible applications in such a short overview, nor to at least roughly complete the whole. I stayed on trying to open and highlight only part of the issue.

During the first period of the corona virus epidemic, in March and April 2020, the second chapter of this book (Formalism) was completed, but all the texts of the two months later printed book "Minimalism of Information" were finished. Then, from March to May, I had a lot of articles and hesitations about their choice, and the decision fell on printing the chronologically first, scrapping some and keeping the works exhibited here (for now, 2020) only in digital format.

What is missing here would be additions to classical information theory (Markov chains and processes in general, ergodic theory, coding), detailed explanations of physical forces or a standard particle model, elaboration of pseudo-information (part of information that remains outside the given perception) and many current topics that are beyond the exact sciences. I might publish some of it myself.

The author at the promotion of the book "Physical Information" in the National and University Library of the Republic of Srpska Banja Luka, December 13, 2019.

Bibliografija

[1] Rastko Vuković: *INFORMATION STORIES* – of physics, biology, law; Economic Institute Banja Luka, 2021. (`https://archive.org/details/stories_202008`)

[2] Rastko Vuković: *MINIMALISM of INFORMATION* – physical information and applications; Economic Institute Banja Luka, 2020. (`https://www.scribd.com/document/421999486/Minimalism-of-Information`)

[3] Rastko Vuković: *PHYSICAL INFORMATION* – in social phenomena and mathematics; Economic Institute Banja Luka, 2019. (`https://archive.org/details/physicalinformation`)

[4] Rastko Vuković: *QUANTUM MECHANICS* – the representation of continuum, of metrical and vector spaces; Economic Institute Banja Luka, 2019. (`https://www.scribd.com/document/377641503/Quantum-Mechanics`)

[5] Rastko Vuković: *MULTIPLICITIES* – Choices for animate and inanimate; Economic Institute Banja Luka, 2018. (`https://archive.org/details/Multiplicity_Rastko`)

[6] Rastko Vuković: *SPACE-TIME* – principles physics of chances; Economic Institute Banja Luka, 2017.(`https://archive.org/details/Principles_Vukovic`)

[7] Rastko Vuković: *Complex Numbers* – Geometry and Algebra of Quaternions; February 10, 2016. (`https://www.academia.edu/21767806/Complex_Numbers_Geometry_and_Algebra_of_Quaternions`)

[8] Rastko Vuković: *TRIANGULATION*, September 13, 2014; (`https://www.academia.edu/8314717/Triangulation`)

[9] Rastko Vukovic: *Quaternion fields* – The six dimensions of space-time; December 3, 2014. (`https://www.gsjournal.net/Science-Journals-Papers/Author/1601/Rastko,%20Vukovic`)

[10] Jean-Pierre Gazeau: *Coherent States* in Quantum Physics, Wiley-VCH, Berlin, 2009. (`http://www.fulviofrisone.com/attachments/article/426/Coherent%20States%20in%20Quantum%20Physics.pdf`)

[11] Arieh Ben-Naim: *On the So-Called Gibbs Paradox, and on the Real Paradox*; Department of Physical Chemistry, The Hebrew University of Jerusalem, Jerusalem, Israel; Received: 12 July 2007 / Accepted: 17 September 2007 / Published: 21 September 2007. (`https://www.mdpi.com/1099-4300/9/3/132/htm`)

[12] Joseph A. Gallian: *Contemporary Abstract Algebra* (EIGHTH EDITION, 2013), University of Minnesota Duluth, 2012938179, 2006. (`file:///C:/Users/Rastko/Downloads/Contemporary%20Abstract%20Algebra%20%20-%20Gallian,%20Joseph%20A._4975.pdf`)

[13] David Pole: *Linear Algebra* – A Modern introduction, Second Edition, Trent University, 2006.

[14] Viktor Prasolov: *PROBLEMS IN PLANE AND SOLID GEOMETRY*, translated and edited by Dimitry Leites.

[15] Slobodan Aljančić: *UVOD U REALNU I FUNKCIONALNU ANALIZU* (treće izdanje); Univerzitet u Beogradu; Izdavačko preduzeće Građevinska knjiga Beograd, 1976.

[16] Svetozar Kurepa: KONAČNO DIMENZIONALNI *VEKTORSKI PROSTORI* I PRIMJENE (drugo izdanje), Zagreb 1976.

[17] Irving Kaplansky: *SET THEORY AND METRIC SPACES*, University of Chicago, Allyn and Bacon, Inc. Boston, 1972.

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